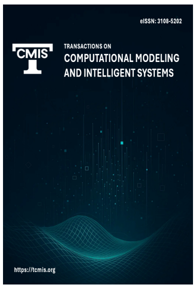

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RESEARCH ARTICLE

Theoretical advances in two-step iterative schemes for nonlinear problems

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Abstract

This study develops a two-step iterative method for solving nonlinear equations, focusing on its convergence properties. Using Taylor series expansion, the method assumes the nonlinear function is at least four times continuously differentiable with bounded derivatives, though this can be restrictive for unbounded higher-order derivatives. To overcome this, a convergence radius is introduced, allowing initial approximations without exact root knowledge. This expands the convergence region without extra smoothness assumptions. Ball convergence analysis establishes error bounds and uniqueness, while local convergence analysis ensures stability and robustness near the root. By combining these perspectives, the framework enhances the theoretical foundation and extends applicability to complex nonlinear problems.

Keywords: Local convergence, Taylor series, computational time, nonlinear problems.

1. Introduction

The problem of determining the solution of non-linear equations is of fundamental importance across various branches of science and engineering. In this work, we focus on approximating the unique solution ¢ to the non-linear equation

$$g(t) = 0, \tag{1}$$

where g is a differentiable function defined on a convex subset $T \subset \Theta$ with $\Theta = \mathbb{R}$ or $\mathbb{C}$. Non-linear equations of this nature frequently arise in a variety of applications [1–3]. Because these equations can be difficult to solve exactly, we usually use an iterative method to find approximate solutions. Among these methods, Newton's method is one of the most basic and widely used.

To analyze the convergence behavior of the proposed two-step method, we utilize Taylor series expansion under the assumption that the nonlinear function g is at least four times continuously differentiable. Additionally, it is assumed that g and its derivatives up to the fourth order remain bounded on the domain Θ. These assumptions ensure that the theoretical foundation of the method is mathematically rigorous and allow us to derive accurate estimates for the order of convergence. To address the existing research gap, we note that classical iterative schemes for solving nonlinear problems often face limitations such as slow convergence, difficulty in handling multiple roots, and restricted regions of convergence. In contrast, the proposed two-step iterative scheme offers significant theoretical advances by explicitly estimating and enlarging the radius of convergence, ensuring faster and more reliable convergence. The key limitations of existing approaches and how our method overcomes them can be summarized as follows:

- **Limited convergence region:** Classical methods converge only locally. *Our method provides an analytically estimated, enlarged radius of convergence.*
- **High computational cost:** Existing methods often require excessive function evaluations. *Our two-step approach reduces iterations and function evaluations while preserving accuracy.*
- **Lack of theoretical guarantees:** Many methods lack rigorous convergence proofs. *We establish formal convergence theorems and provide illustrative examples.*

These improvements demonstrate the effectiveness of the proposed scheme in addressing critical challenges in nonlinear root-finding problems. By imposing these smoothness and boundedness conditions, we are able to demonstrate that the proposed scheme attains faster convergence compared to classical approaches. At the same time, these requirements introduce inherent limitations, since they restrict the class of nonlinear functions to which the method can be directly applied. If the target function exhibits irregularities, discontinuities, or unbounded higher-order derivatives, the convergence rate may deteriorate significantly. In fact, such cases could even lead to divergence, making the method less reliable in highly nonlinear or ill-conditioned scenarios. Another drawback is that verifying boundedness conditions for practical engineering and biomedical problems is often non-trivial. Therefore, while the theoretical results provide valuable insights into the method's performance, caution must be exercised when applying the scheme to real-world systems. The main limitations of these assumptions and their implications for general applicability are explained in detail below:

(1) Consider the real-valued function $g : \Theta \subseteq \mathbb{R} \to \mathbb{R}$ defined on $\Theta = [-0.5, 1.5]$ as which is defined by,

$$g(t) = \begin{cases} t^5 - t^4 + t^3 ln(t^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is evident that g''' is unbounded on Θ, and hence the hypothesis required for convergence analysis is violated. However, g' and g'' remian bounded on Θ, as

$$\lim_{t \to\ 0} t^2 ln(t^2) = \lim_{t \to\ 0} t ln(t^2) = 0.$$

Despite g being continuous, its third derivative is not bounded, which makes it difficult to directly estimate the order of convergence for the proposed method. To overcome this issue, Sections 3 and 4 introduce techniques that avoid relying on higher-order derivatives.

(2) The number of iterations required to achieve a specified accuracy cannot be predetermined.

(3) The radius of convergence is a critical issue in multi-point iterative methods, but it is often not given enough attention.

(4) Most existing methods in the literature do not specify how close the initial approximation must be to the actual root of ¢ in order to achieve the desired order of convergence.

A key tool in this area is local convergence analysis [4–8], which evaluates how an iterative method behaves when the initial guess is close to the actual root. This type of analysis offers valuable insights into two critical aspects: the radius of convergence, which defines the region around the true solution where the method is guaranteed to converge, and speed and stability of convergence, which help determine how efficiently and reliably the method performs.

In order to overcome some of the main drawbacks of iterative approaches, this study introduces a convergence radius that directs the choice of initial approximations even in situations when the precise solution is unknown. By expanding the convergence region, the approach enhances applicability without requiring stronger smoothness or higher-order differentiability requirements. Ball convergence analysis, which offers error bounds, maintains iteration stability, and guarantees the uniqueness of the solution inside the designated region, is used by the framework to demonstrate reliability. Furthermore, local convergence analysis is used to reinforce the theoretical underpinning and increase the resilience of the iterative strategy. These contributions improve the flexibility, efficiency, and practicality of root-finding methods in a variety of nonlinear problems.

The structure of the paper is organized as follows: Section 2 presents the development of the proposed scheme. Section 3 discusses concrete cases of the two-step fourth-order iterative method. Sections 4 and 5 analyze the local convergence conditions and main theorem in detail. Section 6 compares the proposed method with existing two-step methods through numerical experiments demonstrating solution uniqueness, and Section 7 concludes the paper..

2. Methodology

This section introduces a new fourth-order iterative method developed to solve nonlinear equations efficiently. The method consists of two main steps: the first step uses the well-known Newton-Raphson method, and the second step incorporates a weight function. According to the Kung-Traub conjecture, an optimal method without memory, using n function evaluations, can achieve at most an order of 2^{n-1}. The *efficiency index* is a measure of the computational performance of an iterative method, calculated as $E = p^{1/m}$, where p is the order of convergence and m is the number of function evaluations per iteration. The proposed method achieves fourth-order convergence with only three function evaluations per iteration, making it optimal. A key novelty of the proposed approach lies in the use of weight functions within the iterative framework. Unlike prior classical methods, which typically rely on fixed correction formulas, the weight functions here adaptively enhance convergence by dynamically adjusting the contribution of intermediate terms. This innovation not only improves the efficiency index but also strengthens the radius of convergence, setting the proposed method apart from existing high-order schemes. The proposed method achieves fourth-order convergence with only three function evaluations per iteration, making it optimal. The two-step fourth-order iterative method is given below

$$\begin{aligned} y_m &= t_m - \frac{g(t_m)}{g'(t_m)}, \\ t_{m+1} &= t_m - W(\alpha_m) - S(\beta_m) \times \frac{g(y_m)^2\left(2g(t_m) + g(y_m)\right)}{g(t_m)^2 g'(t_m)}, \end{aligned} \tag{2}$$

where,

$$\alpha_m = \frac{g(t_m) + g(y_m)}{g'(t_m)}, \beta_m = \frac{g(t_m)}{g'(t_m)}.$$

The following theorem proves that the proposed method (2) attains fourth-order convergence.

Theorem 1. *Let $\text{¢} \in \widehat{I}$ be a simple root of a real-valued differentiable function $g : \widehat{I} \subseteq \mathbb{R} \to \mathbb{R}$, where $\widehat{I}$ is an open interval. Suppose that the initial approximation t_0 is sufficiently close to ¢, and that the weight functions $W(\alpha_m)$ and $S(\beta_m)$ satisfies the following conditions:*

$$W(0) = 0, W'(0) = 1, W''(0) = W'''(0) = 0, S(0) = 1.$$

Under these conditions, the two-step iterative method defined in (2) achieves fourth-order convergence. Moreover, the error satisfies the following relation

$$\epsilon_{m+1} = (4d_2^3 - d_2d_3 - \frac{1}{24}W^{iv}(0) - 2S'(0)d_2^2)\epsilon_m^4 + O(\epsilon_m^5), \tag{3}$$

where $\epsilon_m = t_m - \text{¢}$ is the the error at the m^{th} iteration.

Proof. Suppose that ¢ be a simple root of the equation $g(t) = 0$, and let $d_i = \frac{g^i(\text{¢})}{i!g'(\text{¢})}$. Applying the Taylor series expansion around ¢, we obtain the following expression

$$g(t_m) = g'(\text{¢})(\epsilon_m + d_2\epsilon_m^2 + d_3\epsilon_m^3 + d_4\epsilon_m^4) + O(\epsilon_m^5). \tag{4}$$

Similarly, the expansion of $g'(t_m)$ yields

$$g'(t_m) = g'(\text{¢})(1 + 2d_2\epsilon_m + 3d_3\epsilon_m^2 + 4d_4\epsilon_m^3) + O(\epsilon_m^4). \tag{5}$$

Using equations (4) and (5), we get the following factor

$$\frac{g(t_m)}{g'(t_m)} = \epsilon_m - d_2\epsilon_m^2 + (2d_2^2 - 2d_3)\epsilon_m^3 + (-4d_2^3 + 7d_2d_3 - 3d_4)\epsilon_m^4 + O(\epsilon_m^5). \tag{6}$$

With the help of Taylor's expansion, we get the error term of the first sub-step of the iterative method (2)

$$y_m - ¢ = d_2\epsilon_m^2 + \left(-2d_2^2 + 2d_3\right)\epsilon_m^3 + \left(4d_2^3 - 7d_2d_3 + 3d_4\right)\epsilon_m^4 + O(\epsilon_m^5) \tag{7}$$

Furthermore, expanding $g(y_m)$ gives

$$g(y_m) = d_2\epsilon_m^2 + \left(-2d_2^2 + 2d_3\right)\epsilon_m^3 + \left(5d_2^3 - 7d_2d_3 + 3d_4\right)\epsilon_m^4 + O(\epsilon_m^5). \tag{8}$$

By using equations (4), (5) and (8), we derive the expression for α_m

$$\frac{g(t_m) + g(y_m)}{g'(t_m)} = \epsilon_m - 2d_2^2\epsilon_m^3 + \left(9d_2^3 - 7d_2d_3\right)\epsilon_m^4 + 0(\epsilon_m^5). \tag{9}$$

Again, by using the Taylor series, the error in the second step of the iterative method (2) is as follows

$$\begin{aligned} t_{m+1} - ¢ = {} & W(0) + \left(1 - W'(0)\right)\epsilon_m - \frac{1}{2}W''(0)\epsilon_m^2 + \left(2W'(0)d_2^2 - \frac{1}{6}W^{(3)}(0) - 2S(0)d_2^2\right)\epsilon_m^3 \\ & + \big(- 9W'(0)d_2^3 + 7W'(0)d_2d_3 + 2W''(0)d_2^2 - \frac{1}{24}W^{(4)}(0) + 13S(0)d_2^3 \\ & - 8S(0)d_2d_3 - 2S'(0)d_2^2\big)\epsilon_m^4 + O(\epsilon_m^5) \end{aligned} \tag{10}$$

We use the following conditions of weight functions

$$W(0) = 0, W'(0) = 1, W''(0) = W'''(0) = 0, S(0) = 1,$$

and get the final error expression

$$\epsilon_{m+1} = (4d_2^3 - d_2d_3 - \frac{1}{24}W^{iv}(0) - 2S'(0)d_2^2)\epsilon_m^4 + O(\epsilon_m^5). \tag{11}$$

This result aligns with the Kung and Traub conjecture, which states that an iterative method without memory, employing n function evaluations per iteration, can achieve at most an order of convergence 2^{n-1}. Since the proposed method requires only three function evaluations per iteration and attains fourth-order convergence, it is optimal in the sense of Kung and Traub's criteria. ☐

3. Particular cases of the proposed two-step method

In this section, we present several specific forms of weight functions $W(\alpha_m)$ and $S(\beta_m)$ that satisfy the conditions established in Theorem (1). The proposed scheme lies in the incorporation of *weight functions* within the iterative formulation. Unlike prior eighth-order methods, which primarily rely on fixed coefficients or standard correction terms, the weight functions dynamically adjust the contribution of intermediate approximations at each iteration. This allows the method to achieve higher accuracy and improved stability, while maintaining optimal computational efficiency. The flexibility offered by these weight functions distinguishes our approach from existing eighth-order schemes and contributes to enhanced convergence behavior, particularly for nonlinear problems with multiple or closely spaced roots. These particular cases demonstrate how the choice of weight functions can ensure the method's fourth-order convergence. By exploring these cases, we not only validate the theoretical results but also provide practical implementations that ensure computational efficiency. Cases are given below:

(1) Let the weight function be defined as

$$W(\alpha_m) = \alpha_m + \frac{1}{24}\alpha_m^4, \quad S(\beta_m) = 1 + \beta_m.$$

Substituting these into the iterative method (2) yields the following two-step fourth-order method (denoted as SK1)

$$\begin{aligned} y_m &= t_m - \frac{g(t_m)}{g'(t_m)}, \\ t_{m+1} &= t_m - \left(\alpha_m + \frac{1}{24}\alpha_m^4\right) - (1+\beta_m) \times \frac{g(y_m)^2\,(2g(t_m)+g(y_m))}{g(t_m)^2 g'(t_m)}. \end{aligned} \tag{12}$$

(2) Another weight function is defined as

$$W(\alpha_m) = \frac{\alpha_m}{1+\alpha_m^3}, \quad S(\beta_m) = 1.$$

Substituting these into the iterative method (2) yields the following two-step fourth-order method (denoted as SK2)

$$\begin{aligned} y_m &= t_m - \frac{g(t_m)}{g'(t_m)}, \\ t_{m+1} &= t_m - \left(\frac{\alpha_m}{1+\alpha_m^3}\right) - \frac{g(y_m)^2\,(2g(t_m)+g(y_m))}{g(t_m)^2 g'(t_m)}. \end{aligned} \tag{13}$$

(3) Another weight function is defined as

$$W(\alpha_m) = \alpha_m, \quad S(\beta_m) = \frac{1}{1+\beta_m}.$$

Substituting these into the iterative method (2) yields the following two-step fourth-order method (denoted as SK3)

$$\begin{aligned} y_m &= t_m - \frac{g(t_m)}{g'(t_m)}, \\ t_{m+1} &= t_m - \alpha_m - \left(\frac{1}{1+\beta_m}\right) \times \frac{g(y_m)^2\,(2g(t_m)+g(y_m))}{g(t_m)^2 g'(t_m)}. \end{aligned} \tag{14}$$

4. Local convergence analysis

Local convergence analysis is critical for understanding iterative methods' behavior at the root of a nonlinear problem. It not only gives a theoretical foundation for establishing convergence, but it also specifies the conditions under which iterations are guaranteed to approach the genuine solution. It determines the region of dependable convergence by assessing the influence of initial assumptions within a defined neighborhood of the root. It also emphasizes how the smoothness of the function and its derivatives affects the order of convergence and error boundaries. Overall, local convergence analysis assures that iterative approaches are stable and predictable in actual applications. In this section, we perform a detailed local convergence analysis of the proposed iterative methods (12), (13), and (14), establishing the criteria for convergence and evaluating the method's effectiveness based on these conditions.

Let $\delta(u,\phi)$ and $\overline{\delta}(u,\phi)$ denote the open and closed balls, respectively, centered at a point $u \in \Theta$ with radius $\phi > 0$, where Θ is an open subset of $\mathbb{R}$. Furthermore, we introduce two positive real constants $\lambda_0 > 0$ and $\lambda > 0$, subject to the constraint $\lambda_0 \leq \lambda$, which hold for all $t, y \in \Theta$.

$$g(¢) = 0, (g')^{-1} \neq 0, \tag{15}$$

$$|(g')^{-1}(g'(t) - g'(¢))| \leq \lambda_0 |t - ¢|, \quad \forall \quad t \in \Theta, \tag{16}$$

$$|(g')^{-1}(g'(t) - g'(y))| \leq \lambda |t - y|, \quad \forall \quad t, y \in \delta(¢, \frac{1}{\lambda_0}). \tag{17}$$

In most cases, the following assumptions can be used,

$$|(g')^{-1} g'(t)| \leq u_1, \quad \forall \quad t \in \delta(¢, \frac{1}{\lambda_0}), u_1 > 1, \tag{18}$$

$$|(g')^{-1} g(y)| \leq u_2 |y - ¢|, \quad \forall \quad u_2 > 1. \tag{19}$$

Lemma: Let $g : \Theta \subseteq \mathbb{R} \to \mathbb{R}$ be a non-linear, continuously differentiable function on an open interval Θ. Suppose $¢ \in \Theta$ is a point where $g'(¢) \neq 0$, and assume that there exists a constant $\lambda_0 > 0$ such that (16) holds. Then for any $t \in \Theta$ and any $\lambda \in [0, 1]$, the following estimates hold:

$$|(g')^{-1}g'(t)| \leq 1 + \lambda_0 |t - ¢|, \tag{20}$$

$$|(g')^{-1}g'(¢ + \gamma(t - ¢))| \leq 1 + \lambda_0 |t - ¢|, \tag{21}$$

$$|(g')^{-1}g(t)| \leq (1 + \lambda_0 |t - ¢|)|t - ¢|. \tag{22}$$

Proof. To establish the stated inequalities, we proceed by verifying each bound individually.

(1) Proof of inequality (20).
Under the assumption (20), we have

$$\left|(g')^{-1}g'(t)\right| = \left|1 + (g')^{-1}(g'(t) - g'(¢))\right| \leq 1 + \left|(g')^{-1}(g'(t) - g'(¢))\right| \leq 1 + \lambda_0 |t - ¢|.$$

(2) Proof of inequality (21) Let $t = ¢ + \gamma(t - ¢)$ for $\gamma \in [0, 1]$. Then

$$\left|(g')^{-1}g'(t)\right| = |(g')^{-1}g'(¢ + \gamma(t - ¢))| \leq 1 + \lambda_0 \gamma |t - ¢| \leq 1 + \lambda_0 |t - ¢|.$$

(3) Proof of inequality (22)
By the mean value theorem, there exists $\gamma \in (¢, t)$ such that

$$g(t) - g(¢) = g'(\gamma)(t - ¢).$$

Assume that $g(¢) = 0$, we get

$$g(t) = g'(\gamma)(t - ¢),$$

and thus

$$\left|(g')^{-1}g(t)\right| = |(g')^{-1}g'(\gamma)| \times |t - ¢| \leq (1 + \lambda_0 |t - ¢|)|t - ¢|.$$

□

5. Local convergence conditions

To analyze the local convergence, we define a specific scalar function and parameters that help to determine the behavior of the iterative method. Let us define the scalar function $\eta_1(k)$ defined on the interval $T = \left[0, \frac{1}{\lambda_0}\right)$ as follows

$$\eta_1(k) = \frac{\lambda k}{2(1 - \lambda_0 k)}.$$

Define the parameter,

$$r_1 = \frac{2}{2\lambda_0 + \lambda} < \frac{1}{\lambda_0},$$

which satisfies $r_1 < \frac{1}{\lambda_0}$. It is evident that $\eta_1(r_1) = 1$ and the inequality $0 \leq \eta_1(k) < 1$ holds for all $k \in [0, r_1)$. Next, define a second scalar function $\eta_2(k)$ over the interval $[0, r_1)$, given by

$$\eta_2(k) = k + W\left[\frac{u_1 + u_2\eta_1(k)}{1 - \lambda_0 k}\right] + S\left[\frac{u_1}{1 - \lambda_0 k}\right]\left(\frac{1}{1 - u_1 k}\right)^2 \times \left(\frac{1}{1 - \lambda_0 k}\right) \times ((u_2\eta_1(k)) \times (2u_1 + u_2\eta_1(k)).$$

Define its corresponding auxiliary function as $p_1(k) = \eta_2(k) - 1$. Under the assumption

$$\eta_2(0) = W(0) < 1,$$

we have $p_1(0) < 0$, and since $p_1(k) < \infty$ as $k \to r_1^-$, continuity implies the existence of at least one root of $p_1(k)$ in $(0, r_1)$. Denote the smallest such root by r_2.
The convergence radius r for the iterative method (2) is then defined as

$$r = min\{r_i, i = 1, 2\}. \tag{23}$$

Therefore, for all $k \in [0, r)$, the following bounds hold:

$$0 \le \eta_1(k) < 1, \tag{24}$$

$$0 \le p_1(k) < 1, \quad \text{for each} \quad k \in [0, r). \tag{25}$$

Theorem 2. *Let $M : \Theta \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable non-linear function satisfying conditions (15)-(17). Suppose that $¢ \in \Theta$ is a simple root of M, and let the initial approximation $t_0 \in \delta(¢, r) = t \in \mathbb{R} : |t - ¢| < r$. Then the sequence t_m generated by the iterative method (2) is well-defined and converges to ¢.*

In addition, for all $m \ge 0$, the following bounds hold:

$$|y_m - ¢| \le \eta_1(|t_m - ¢|)|t_m - ¢| < |t_m - ¢| < r, \tag{26}$$

$$|t_{m+1} - ¢| \le \eta_2(|t_m - ¢|)|t_m - ¢| < |t_m - ¢| < r, \tag{27}$$

where η_1 and η_2 are non-negative scalar functions defined based on the properties of M and its derivative.

Furthermore, if the parameter R is chosen such that

$$R \in \left[r, \frac{2}{\lambda_0}\right),$$

then the closed ball $\overline{\delta}(¢, r) \subseteq \Theta$, and the root ¢ is the unique solution of $g(t) = 0$ within the ball.

Proof. We verify the validity of bounds (26) and (27) using mathematical induction. Suppose that the initial point $t_0 \in \delta(¢, r)$. We begin by showing that $g'(t_0)$ is invertible. From inequality (16), it follows that

$$|I - (g')^{-1}g'(t_0)| = |(g')^{-1}(g'(t_0) - g'(¢))| \le \lambda_0|t_0 - ¢| < 1. \tag{28}$$

Since the right-hand side is strictly less than one, Banach's lemma [9] ensures that $g'(t_0)$ is invertible. Consequently, we obtain the following estimate

$$|g'(t_0)^{-1}g'(¢)| \le \frac{1}{1 - \lambda_0|t_0 - ¢|}. \tag{29}$$

This confirms that t_0 is well-defined within the iterative process. To further analyze the method, we express the difference $g(t) - g(¢)$ using the integral form of the mean value theorem [10]

$$g(t_0) = g(t_0) - g(¢) = \int_0^1 g'(¢ + \gamma(t_0 - ¢))(t_0 - ¢)d\gamma. \tag{30}$$

Since $|¢ + \gamma(t_0 - ¢) - ¢| = \gamma|t_0 - ¢| < r$, it follows that $¢ + \gamma(t_0 - ¢) \in \delta(¢, r)$ for all $\gamma \in [0, 1]$. Therefore, substituting (30) into the scaled form yields

$$(g')^{-1}g(t_0) = \int_0^1 g'^{-1}g'(¢ + \gamma(t_0 - ¢))(t_0 - ¢)d\gamma. \tag{31}$$

We now analyze the first sub-step of the iterative method (2) at $m = 0$

$$\begin{aligned} y_0 - ¢ &= t_0 - ¢ - g'(t_0)^{-1}g(t_0), \\ &= -g'(t_0)^{-1}(g(t_0) - g'(t_0)(t_0 - ¢)). \end{aligned} \tag{32}$$

Taking absolute on both sides of equation (29) and applying inequalities (17), (24), (26), and (28), we derive the following estimate

$$\begin{aligned} |y_0 - ¢| &= \left| -g'(t_0)^{-1}g'(¢) \int_0^1 g'^{-1}\left(g'(¢ + \gamma(t_0 - ¢)) - g'(t_0)\right)(t_0 - ¢)\,d\gamma \right|, \\ &\le \left|g'(t_0)^{-1}g'(¢)\right| \left| \int_0^1 g'^{-1}\left(g'(¢ + \gamma(t_0 - ¢)) - g'(t_0)\right)(t_0 - ¢)\,d\gamma \right|, \\ &\le \frac{\lambda|t_0 - ¢|}{2(1 - \lambda_0|t_0 - ¢|)}|t_0 - ¢| \le \eta_1(|t_0 - ¢|)|t_0 - ¢| < |t_0 - ¢| < r. \end{aligned} \tag{33}$$

This confirms that the bound (33) is satisfied for $m = 0$, implying that $t_0 \in \delta(\text{¢}, r)$. Thus, the first iterate t_0 is well-defined and lies within the neighborhood of ¢. We now proceed to verify that the second sub-step of the iterative method (2), ensuring that is well-defined and satisfies the conditions required for invertibility.

To prove $g(t_0) \neq 0$, we use the inequality (22)

$$|g'^{-1}g(t_0)| \le (1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| \le d_1(|t_0 - \text{¢}|) < 1. \tag{34}$$

Then $(g'(t_0))^{-1}$ exists, and

$$|g(t_0)^{-1}g'(\text{¢})| \le \frac{1}{1 - d_1(|t_0 - \text{¢}|)}. \tag{35}$$

Using the inequalities (17), (18), (19) and (27) to express the first variable weight function at $m = 0$

$$\begin{aligned}|\alpha_0| &= |g'(t_0)^{-1}g(\text{¢})||g'(t_0)^{-1}(g(t_0) + g(y_0))|,\\ &\le \frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2|y_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|},\\ &\le \frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2\eta_1(|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\end{aligned} \tag{36}$$

and the second variable weight function at $m = 0$ is

$$\beta_0 = |g'(t_0)^{-1}g'(\text{¢})||g'(t_0)^{-1}g(t_0)| \le \frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}.$$

By using the inequalities (5) and (37), to get the following estimates $|W(\alpha_0)|$ and $S(\beta_0)$

$$|W(\alpha_0)| \le W\left[\frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2\eta_1(|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\right], \tag{37}$$

and

$$|S(\beta_0)| \le S\left[\frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\right]. \tag{38}$$

Thus (37) and (38) shows that α_0 and β_0 are well-defined. Now, we can write the second sub-step of the two-step iterative method (2) for $m = 0$

$$\begin{aligned}t_1 - \text{¢} = t_0 - \text{¢} - W(\alpha_0) - S(\beta_0) \times ((g(t_0)^{-1}g'^2((g'(t_0))^{-1}g'(\text{¢}))\\ (g'(t_0)^{-1}g(y_0))^2(g'(t_0)^{-1}(2g(t_0) + g(y_0)))),\end{aligned}$$

Taking assertions (18), (19), (28), (34),(37), (38) and absolute on both sides, we obtain the following second-step expression

$$\begin{aligned}|t_1 - \text{¢}| \le |t_0 - \text{¢}| &+ W\left[\frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2\eta_1(|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\right]\\ &+ S\left[\frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\right] \times \left(\frac{1}{1 - d_1|t_0 - \text{¢}|}\right)^2 \times \left(\frac{1}{1 - \lambda_0|t_0 - \text{¢}|}\right)\\ &\times ((u_2|y_0 - \text{¢}|)^2 \times (2(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2|y_0 - \text{¢}|)),\\ &\le |t_0 - \text{¢}| + W\left[\frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2\eta_1(|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\right]\\ &+ S\left[\frac{(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}|}{1 - \lambda_0|t_0 - \text{¢}|}\right] \times \left(\frac{1}{1 - d_1|t_0 - \text{¢}|}\right)^2 \times \left(\frac{1}{1 - \lambda_0|t_0 - \text{¢}|}\right)\\ &\times ((u_2\eta_1(|t_0 - \text{¢}|)|t_0 - \text{¢}|)^2 \times (2(1 + \lambda_0|t_0 - \text{¢}|)|t_0 - \text{¢}| + u_2\eta_1(|t_0 - \text{¢}|)|t_0 - \text{¢}|).\end{aligned} \tag{39}$$

This establishes the validity of inequality (39), confirming that $t_1 \in \delta(\text{¢}, r)$. By applying the principle of mathematical induction and replacing t_0, y_0, t_1 with t_m, y_m, t_{m+1}, we conclude that inequalities (25) and (26) hold for all $m \in \mathbb{R}$.

Since $|t_{m+1} - \cent| \leq |\cent - \cent| < r$, each iterate remains with the neighborhood $\delta(\cent, r)$, preserving the assumptions required by the convergence analysis. Additionally, given that the function $\eta_2(k)$ is increasing, we can recursively estimate

$$\begin{aligned} |t_{m+1} - \cent| &\leq \eta_2(t)|t_m - \cent| \leq \eta_2(t)\eta_2(|t_{m-1} - \cent|)|t_{m-1} - \cent|, \\ &\leq \eta_2(t)^2\eta_2(|t_{m-2} - \cent|)|t_{m-2} - \cent| \leq \cdots \leq \eta_2(t)^{n+1}|t_0 - \cent|. \end{aligned}$$

Taking the limit as $k \to \infty$, and assuming $\eta_2(k)^{n+1} \to 0$, we obtain

$$\lim_{t\to\infty} |t_m - \cent| = 0, \implies \lim_{t\to\infty} t_m = \cent.$$

Hence, the iterative method (2) converges to the unique solution $\cent$ of the non-linear equation. To prove the uniqueness, suppose there exists another solution $\psi^* \in \overline{\delta}(\cent, R)$ such that $g(\psi^*) = 0$. Define the expression

$$\Omega = \int_0^1 g'^{*} + \gamma(\cent - \psi^*))d\gamma.$$

Since $g(\cent) = 0$, applying the mean value form gives

$$g(\psi^*) - g(\cent) = \Omega(\psi^* - \cent) = 0.$$

To show that Ω is invertible, consider the following estimate

$$\begin{aligned} \left|(g')^{-1}(\Omega - g'(\cent))\right| &\leq \left|\int_0^1 \lambda_0(\psi^* + \gamma(\cent - \psi^*) - y^*)\, d\gamma\right|, \\ &\leq \lambda_0 \int_0^1 (1-\gamma)|\psi^* - \cent|\, d\gamma \leq \frac{\lambda_0}{2} R < 1. \end{aligned} \tag{40}$$

Since the expression is strictly less than 1, Banach's lemma guarantees that Ω is invertible. Therefore, from $\Omega(\psi^* - \cent) = 0$, it follows that $\psi^* = \cent$. This completes the proof of uniqueness. □

To illustrate the applicability of the general convergence theorem (2), we now consider several special cases. Each example is shown to satisfy the key assumptions of the theorem (2), particularly those related to the convergence radius. While the first sub-step of the iterative method (2) remains fixed, the second step is expressed in the generalized form (39) enables the inclusion of various weight functions. This unified framework preserves the validity of the convergence results across different iterative schemes.

Method 1:

$$\begin{aligned} |t_1 - \cent| = |t_0 - \cent| + &\left| \left(\frac{g(t_0) + g(y_0)}{g'(t_0)} + \frac{1}{24}\left(\frac{g(t_0) + g(y_0)}{g'(t_0)}\right)^4 \right) - \left(1 + \frac{g(t_0)}{g'(t_0)}\right) \right. \\ &\left. \times \frac{g(y_0)^2\,(2g(t_0) + g(y_0))}{g(t_0)^2 g'(t_0)} \right|, \\ &\leq |t_0 - \cent| + \left(\frac{(1 + \lambda_0|t_0 - \cent|)|t_0 - \cent| + u_2\eta_1(|t_0 - \cent|)|t_0 - \cent|}{1 - \lambda_0|t_0 - \cent|} \right. \\ &\left. + \frac{1}{24}\left(\frac{(1 + \lambda_0|t_0 - \cent|)|t_0 - \cent| + u_2\eta_1(|t_0 - \cent|)|t_0 - \cent|}{1 - \lambda_0|t_0 - \cent|} \right)^4 \right) \\ &+ \left(1 + \frac{(1 + \lambda_0|t_0 - \cent|)|t_0 - \cent|}{1 - \lambda_0|t_0 - \cent|}\right) \times \left(\frac{1}{1 - d_1|t_0 - \cent|}\right)^2 \times \left(\frac{1}{1 - \lambda_0|t_0 - \cent|}\right) \\ &\times ((u_2|y - \cent|)^2 \times (2(1 + \lambda_0|t_0 - \cent|)|t_0 - \cent| + u_2|y - \cent|)), \\ &\leq \eta_2(|t_0 - \cent|)|t_0 - \cent| < |t_0 - \cent| < r. \end{aligned} \tag{41}$$

Method 2:

$$
\begin{aligned}
|t_1-¢| &= |t_0-¢| + \left| \frac{\frac{g(t_m)+g(y_m)}{g'(t_m)}}{1+\left(\frac{g(t_m)+g(y_m)}{g'(t_m)}\right)^3} - \frac{g(y_0)^2\left(2g(t_0)+g(y_0)\right)}{g(t_0)^2 g'(t_0)} \right|, \\
&\leq |t_0-¢| + \left(\frac{\frac{(1+\lambda_0|t_0-¢|)|t_0-¢|+u_2\eta_1(|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}}{1+\left(\frac{(1+\lambda_0|t_0-¢|)|t_0-¢|+u_2\eta_1(|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}\right)^3} \right) + \left(\frac{1}{1-u_1|t_0-¢|}\right)^2 \\
&\times \left(\frac{1}{1-\lambda_0|t_0-¢|}\right) \times \left((u_2|y-¢|) \times (2(1+\lambda_0|t_0-¢|)|t_0-¢| + u_2|y-¢|)\right), \\
&\leq \Bigg[1 + \left(\frac{\frac{(1+\lambda_0|t_0-¢|)|t_0-¢|+u_2\eta_1(|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}}{1+\left(\frac{(1+\lambda_0|t_0-¢|)|t_0-¢|+u_2\eta_1(|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}\right)^3} \right) + \left(\frac{1}{1-u_1|t_0-¢|}\right)^2 \\
&\times \left(\frac{1}{1-\lambda_0|t_0-¢|}\right) \times \left((u_2\eta_1|t_0-¢|)^2|t_0-\eta| \times (2(1+\lambda_0|t_0-¢|)|t_0-¢| + u_2\eta_1|t_0-¢|)\right) \Bigg] \\
&|t_0-¢|, \leq \eta_2(|t_0-¢|)|t_0-¢| < |t_0-¢| < r.
\end{aligned}
\tag{42}
$$

Method 3:

$$
\begin{aligned}
|t_1-¢| &= |t_0-¢| + \Bigg| \left(\frac{g(t_m)+g(y_m)}{g'(t_m)}\right) - \left(\frac{1}{1+\frac{g(t_m)}{g'(t_m)}}\right) \\
&\quad \times \frac{g(y_0)^2\left(2g(t_0)+g(y_0)\right)}{g(t_0)^2 g'(t_0)} \Bigg|, \\
&\quad \leq |t_0-¢| + \left(\frac{(1+\lambda_0|t_0-¢|)|t_0-¢| + u_2\eta_1(|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}\right) \\
&\quad + \left(\frac{1}{1+\frac{(1+\lambda_0|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}}\right) \times \left(\frac{1}{1-u_1|t_0-¢|}\right)^2 \times \left(\frac{1}{1-\lambda_0|t_0-¢|}\right) \\
&\quad \times \left((u_2|y-¢|)^2 \times (2(1+\lambda_0|t_0-¢|)|t_0-¢| + u_2|y-¢|)\right), \\
&\quad \Bigg[1 + \left(\frac{(1+\lambda_0|t_0-¢|)|t_0-¢| + u_2\eta_1(|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}\right) + \left(\frac{1}{1+\frac{(1+\lambda_0|t_0-¢|)|t_0-¢|}{1-\lambda_0|t_0-¢|}}\right) \\
&\quad \times (2(1+\lambda_0|t_0-¢|)|t_0-¢| + (u_2\eta_1(|t_0-¢|))^2|t_0-¢| \\
&\quad \leq \eta_2(|t_0-¢|)|t_0-¢| < |t_0-¢| < r.
\end{aligned}
\tag{43}
$$

6. Numerical results

This section presents the theoretical validation of the proposed two-step iterative method defined in (12)-(14), along with its numerical results for solving non-linear equations. The method's performance is evaluated through computational experiments and compared with existing two-step methods. Numerical results confirm that the method achieves improved convergence radius and enhanced stability for various non-linear functions. All computations were performed using the computer algebra system Maple 18, and the results are presented up to six decimal places for consistency.

For the comparison, we take some existing two-step iterative methods for solving non-linear equations are given below: Delshad et al. [11] investigated the analysis of local convergence of a two-step Kung-Traub type iterative method for solving non-linear equations, with iterations defined for $m = 0, 1, 2, ...$ The method (ADL) is structured as follows

$$\begin{aligned} y_m &= t_m - \frac{g(t_m)}{g'(t_n)}, \\ t_{m+1} &= y_n - H(t_m), \end{aligned} \tag{44}$$

where, the function $H(t_m)$ is given by

$$H(t_m) = \frac{g(t_m)^2 g(y_m)}{g'(t_m)(g(y_m) - g(t_m))^2}.$$

Behl et al. [12] studied the local convergence behavior of a two-step iterative method of fourth-order, commonly referred to as the weighted Newton method. The iterative scheme (BRK) is defined as follows

$$\begin{aligned} y_m &= t_m - \mu \frac{g(t_m)}{g'(t_m)}, \\ t_{m+1} &= y_m - S_m^{-1}(b_1 g(t_m) + b_2 g(y_m)), \end{aligned} \tag{45}$$

where t_0 is the initial approximation and S_m is defined by

$$S_m = v_1 g'(t_m) + v_2 g'\left(\frac{t_m + y_m}{2}\right) + v_3 g'(y_m).$$

The parameters $\mu, b_1, b_2, v_1.v_2, v_3$ are constants chosen to ensure fourth-order convergence. Their expressions are given

$$\begin{aligned} v_1 &= -\frac{1}{3} d_2(3\mu^2 - 7\mu + 2), \quad v_2 = -\frac{4}{3} d_2(2\mu - 1), \\ v_3 &= \frac{1}{3} d_2(\mu - 2), \quad b_1 = -b_2(\mu^2 - \mu + 1), \quad \text{for} \quad \mu, b_2 \neq 0. \end{aligned}$$

This method incorporates a parameterized weighted combination of derivative evaluations to achieve higher-order convergence while maintaining computational efficiency.

Argyros et al. [13] investigated the local convergence analysis of a two-step iterative method (AIK) for $m = 0, 1, 2, \ldots,$

$$\begin{aligned} y_m &= t_m - \mu \frac{g(t_m)}{g'(t_m)}, \\ t_{m+1} &= y_m - \frac{g(y_m)}{g'(t_m)} \times \frac{g(t_m) + \beta g(y_m)}{g(t_m) + (\beta - 2) g(y_m)}. \end{aligned} \tag{46}$$

Example 1: Let $U = Y = \mathbb{R}$ and consider the non-linear function g_1 defined on the closed ball $\Theta = \overline{\delta}(0, 1)$ as follows [14]:

$$g_1(t) = e^t - 1,$$

with the root $¢ = 0$. The corresponding first derivative is given by

$$g'(t) = e^t.$$

Using conditions (15)-(19), we compute the associated constants as

$$\lambda_0 = e - 1, \quad \lambda = u_1 = u_2 = e^{\frac{1}{e-1}}.$$

Numerical computations were performed using the computer algebra system Maple 18. As shown in table (1), the two step iterative methods SK1, SK2, and SK3 demonstrate a significantly wider region of convergence. This extended convergence radius confirms the superior performance and robustness of the proposed method (2), making it applicable to a wider selection of initial approximations.

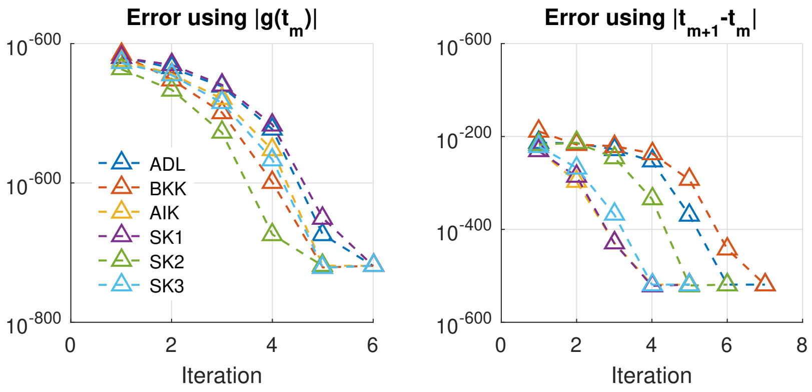

Figure 1. Error plots of the numerical schemes for solving nonlinear equation used in Example 1.

Table 1. Convergence radius analysis of different iterative methods for $g_1(x)$

Method	r_1	r_2	$r = min\{r_1, r_2\}$
SK1	0.382689	0.588528	0.588528
SK2	0.382689	0.586257	0.586257
SK3	0.382689	0.563786	0.563786

Table 2. Numerical outcomes of iterative schemes for solving $g_1(x)$

Method	$\lvert g\left(t_m\right)\rvert$	$\lvert t_{m+1} - t_m\rvert$	$CPU - time$
SK1	0.43e-635	0.875e-435	0.0097
SK2	9.08e-709	0.07e-505	0.0096
SK3	6.76e-803	9.07e-560	0.0086
ADL	3.98e-649	0.98e-434	0.0887
BRK	0.09e-588	0.05e-378	0.0586
AIK	9.076e-586	0.14e-420	0.1075

The numerical outcomes from Tables 1–2 and Figure 1 clearly demonstrate that the newly developed method outperforms the existing approaches in terms of function value error, residual error, and CPU time. This improvement is achieved by selecting the initial guess values according to the assumptions established in the local convergence analysis of Theorem 2, which ensures both efficiency and robustness of the proposed scheme.

Example 2: Consider the non-linear function $g_2(t)$ defined on the interval $\Theta = [1, 4]$ as [15]:

$$g_2(t) = \frac{3}{4}t^{\frac{4}{3}} - t.$$

The exact simple root of this function is $¢ = \left(\frac{4}{3}\right)^3$. The first derivative is given by

$$g'^{\frac{1}{3}} - 1.$$

Using conditions (15)-(19), we compute the associated constants as

$$\lambda_0 = \lambda = 1, u_1 = u_2 = 1.76220315590460.$$

Table (2) shows that the two-step iterative methods SK1, SK2, and SK3 demonstrate a significantly wider region of convergence compared to the methods ADL, BRK, and AIK.

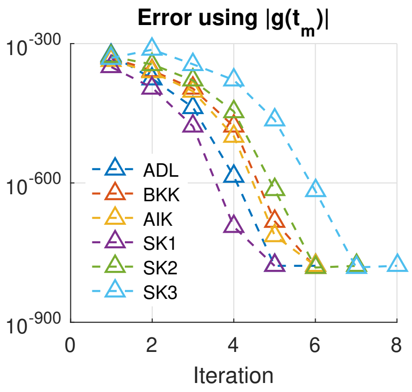

Figure 2. Error plots of the numerical schemes for solving nonlinear equations used in Example 2.

Table 3. Convergence radius analysis of different iterative methods for $g_2(x)$

Method	r_1	r_2	$r = min\{r_1, r_2\}$
SK1	0.6666	0.456792	0.456792
SK2	0.6666	0.453527	0.453527
SK3	0.6666	0.445624	0.445624

Table 4. Numerical outcomes of iterative schemes for solving $g_2(x)$

Method	$\|g(t_m)\|$	$\|t_{m+1} - t_m\|$	$CPU - time$
SK1	0.98e-643	0.875e-476	0.0098
SK2	9.11e-577	0.07e-378	0.0013
SK3	9.09e-789	9.07e-657	0.0034
ADL	5.76e-600	0.98e-398	0.0865
BRK	4.66e-579	0.05e-277	0.0335
AIK	3.44e-409	0.14e-257	0.0975

The numerical findings presented in 3–4 and Figure 2 provide strong evidence that our proposed method significantly outperforms traditional techniques. In particular, it achieves lower function value errors, reduced residual errors, and faster CPU times. These improvements stem from the careful selection of initial guess values guided by the local convergence analysis in Theorem 2, which not only enhances accuracy but also minimizes computational inefficiency often encountered in classical iterative solvers.

Example 3 Returning to the motivational example discussed at the beginning of this paper, the associated parameters are taken as

$$\lambda_0 = \lambda = 96.6629073, u_1 = u_2 = 2.$$

As observed in table (3), the proposed methods SK1, SK2, and SK3 outperform the compared methods by yielding a significantly larger radius of convergence. This result underscores the method's enhanced efficiency and stability in solving non-linear equations.

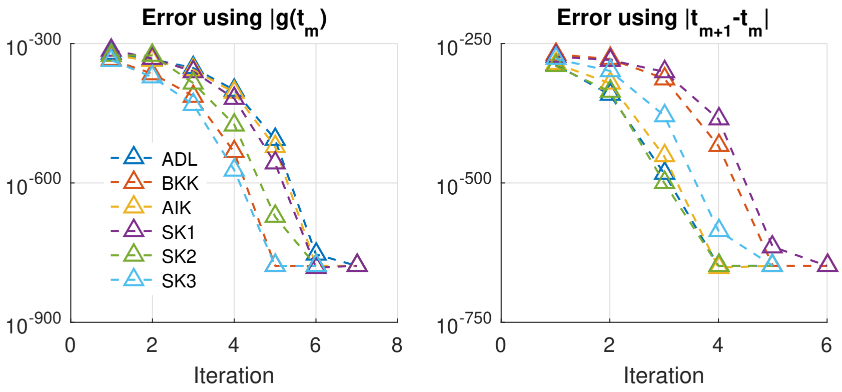

Figure 3. Error plots of the numerical schemes for solving nonlinear equations used in Example 3.

Table 5. Convergence radius analysis of different iterative methods for $g_3(x)$

Method	$\boldsymbol{r_1}$	$\boldsymbol{r_2}$	$\boldsymbol{r = min\{r_1, r_2\}}$
SK1	1.175903	1.134579	1.134579
SK2	1.175903	1.132576	1.132576
SK3	1.175903	1.102549	1.102549

Table 6. Numerical outcomes of iterative schemes for solving $g_3(x)$

Method	$\boldsymbol{\lvert g(t_m)\rvert}$	$\lvert t_{m+1} - t_m\rvert$	$\boldsymbol{CPU - time}$
SK1	0.98e-786	0.85e-587	0.0034
SK2	0.12e-856	0.27e-648	0.0012
SK3	6.98e-879	4.37e-609	0.0055
ADL	3.34e-677	8.98e-587	0.0365
BRK	0.09e-708	4.04e-670	0.0776
AIK	1.63e-754	8.19e-599	0.0875

Tables 5–6 and Figure 3 demonstrate the advantages of the newly devised scheme over existing methods. The method consistently produces less function and residual errors while using less CPU time. This efficiency increase is ascribed to the use of starting estimates selected in accordance with Theorem 2's theoretical framework, which guarantees a trustworthy trade-off between numerical stability and convergence speed. Thus, the approach shows both theoretical validity and useful computational efficiency.

Beyond the three case studies presented, the proposed root-finding framework and iterative schemes have broad applicability across various engineering and applied science domains. Potential applications include solving high-degree nonlinear equations in control systems, modeling complex biochemical networks, optimizing multi-parameter mechanical systems, and addressing fractional-order models in signal processing and biomedical engineering. These extensions demonstrate the versatility and practical relevance of the proposed method across diverse scientific and engineering problems.

7. Conclusion

In this study, a new two-step approach to finding simple roots was developed. To ensure a thorough theoretical foundation, the local convergence analysis has been carried out both with and without derivatives. The numerical results in Tables 1-6 and Figures 1-3 clearly show that the suggested strategy consistently outperforms previous approaches. Specifically, it achieves lower residual errors (computed via function values), requires fewer corrective iterations, exhibits broader convergence radii, and reduces CPU execution time. These advantages establish the method as a strong alternative for solving nonlinear problems efficiently.

The proposed method, while significantly improving convergence and efficiency, has some limitations. These include potential sensitivity to the choice of initial guesses, increased computational cost for extremely high-degree polynomials, and the need for accurate evaluation of function derivatives in certain cases. Future work will focus on addressing these aspects to further enhance robustness and applicability.

Future Directions:

- Extension of the proposed scheme to systems of nonlinear equations.
- Adaptation for parameter-dependent problems in engineering and applied sciences.
- Development of *parallel or GPU-accelerated implementations* for large-scale applications.
- Exploration of hybrid schemes combining the method with machine learning techniques for enhanced convergence prediction.

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Conflict of interest

There is no conflict of interest to disclose.

Author contributions

The study was conducted entirely by the sole author.

Declaration of using AI tools

The authors declare that they have not used any type of generative artificial intelligence for the writing of this manuscript, nor for the creation of images, graphics, tables, or their corresponding captions.

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