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RESEARCH ARTICLE

Analysis of quantum-deformed fractional Schrödinger equations based on generalized Hermite polynomials

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Abstract

We investigate the fractional Schrödinger equation driven by the generalized (q,τ)-Caputo nabla derivative and an energy operator diagonalized by (q,τ)-Hermite polynomials. A modal decomposition approach is adopted, allowing the solution to be expanded in terms of an orthogonal polynomial basis with deformed weights and spectral parameters. We derive a Lyapunov-type energy functional and prove a decay estimate under small initial energy using a generalized (q,τ)-Grönwall inequality. A complete nonlinear analysis is presented, including bounds on quartic interaction terms, global existence of solutions in both L^2 and Sobolev spaces, and a corollary on uniqueness and modal regularity. Furthermore, we formulate a fractional energy balance and provide numerical illustrations demonstrating long-time decay in truncated-mode expansions. These results highlight the stability and regularity of quantum states under (q,τ)-deformed fractional dynamics and show that (q,τ)-Hermite polynomials serve as a powerful analytical tool in modeling memory-dependent quantum systems.

Keywords: Quantum calculus, generalized gamma function, (q,τ)-fractional operators, symmetric special functions, Schrödinger equation

1. Introduction

In the study of special functions, the classical gamma function $\Gamma(\alpha)$ plays a central role in various areas of mathematics and physics. However, in the framework of q-calculus [1–3] and quantum algebra, classical functions are often generalized to their q-analogues (see [4, 5] and for application see [6–11]). The quantum gamma function $\Gamma_{q,\tau}(\alpha)$, a two-parameter generalization, extends the concept of the classical gamma function to accommodate both quantum and deformation parameters, q and τ. This function, defined as [12–15]

$$\Gamma_{q,\tau}(\alpha) = (1-q)^{1-\alpha} \prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+n)}},$$

is of particular interest in various areas of mathematical physics, including the theory of q-special functions, orthogonal polynomials, and the representation theory of quantum groups.

A key application of $\Gamma_{q,\tau}(\alpha)$ is in the study of q-Hermite polynomials, which arise in the context of quantum harmonic oscillators and are essential in quantum mechanics, signal processing, and other areas of mathematical physics. The classical q-Hermite polynomials, denoted $H_n(x|q)$, satisfy recurrence relations and orthogonality conditions similar to their classical counterparts. However, their properties are modified in the quantum setting, where the quantum gamma function plays a crucial role in their normalization and weight functions.
Fractional quantum mechanics has emerged as a powerful framework to model memory effects and nonlocal interactions in quantum systems. The classical Schrödinger equation, when generalized to incorporate fractional time derivatives [16–18], yields equations with profound implications in anomalous transport, decoherence, and quantum control [19–21]. These generalizations typically involve the Caputo or Riemann-Liouville derivatives [22–24]. However, recent advances in discrete and quantum calculus have motivated the use of (q,τ)-deformations, which provide an elegant mechanism for encoding spectral deformation and memory simultaneously.
In this work, we focus on a fractional Schrödinger equation of the form

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t),$$

where ${}^{C}\nabla^{\gamma}_{q,\tau}$ is the Caputo-type (q,τ)-nabla fractional derivative and $\hat{H}_{q,\tau}$ is a self-adjoint operator with spectral basis given by the (q,τ)-Hermite polynomials. These polynomials form a weighted orthogonal system under a deformed inner product, and their associated eigenvalues exhibit nonlinear spectral scaling controlled by the deformation parameters q and τ.
Our motivation is to investigate stability, long-time behavior, and solution regularity in quantum models combining fractional memory with (q,τ)-deformation. The specific contribution is the formulation and analysis of this combined model using Hermite-type modal expansions. The analytical methods adapt established spectral, Caputo-fractional, and energy techniques: we introduce a Lyapunov-type functional $\mathcal{L}(t)$, derive conditional decay estimates through a (q,τ)-Grönwall-type inequality, and estimate nonlinear quartic interactions. Under the stated spectral and kernel assumptions, the modal representation also supports an investigation of regularity in Sobolev-type spaces H^s. Truncated-mode computations illustrate the predicted behavior and the influence of the deformation parameters. Thus, the contribution concerns the particular model and its analysis, rather than new spectral or energy-estimation methods.

2. Quantum computing

Quantum calculus replaces the classical derivative with the q-difference operator defined by:

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1.$$

The corresponding q-integral (Jackson integral) is defined as:

$$\int_0^a f(t)\,d_q t = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n),$$

under suitable convergence conditions. These operators lead to a natural extension of classical analysis, allowing one to define q-analogues of many known functions.
The classical Gamma function is extended in q-calculus by:

$$\Gamma_q(\alpha) = (1-q)^{1-\alpha}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{\alpha+n}}, \quad \alpha > 0, \quad 0 < q < 1.$$

The quantum gamma function $\Gamma_{q,\tau}(\alpha)$ is a two-parameter generalization of the classical gamma function, defined by (see Fig.1):

$$\Gamma_{q,\tau}(\alpha) = (1-q)^{1-\alpha}\prod_{n=0}^{\infty}\frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+n)}}, \quad 0 < q < 1,\ \tau > 0. \tag{1}$$

Fig. 1 illustrates the behavior of the (q,τ)-deformed Gamma function $\Gamma_q(\alpha)$ across a range of fractional orders $\alpha \in (0.1, 5)$ and for selected combinations of deformation parameters (q,τ),

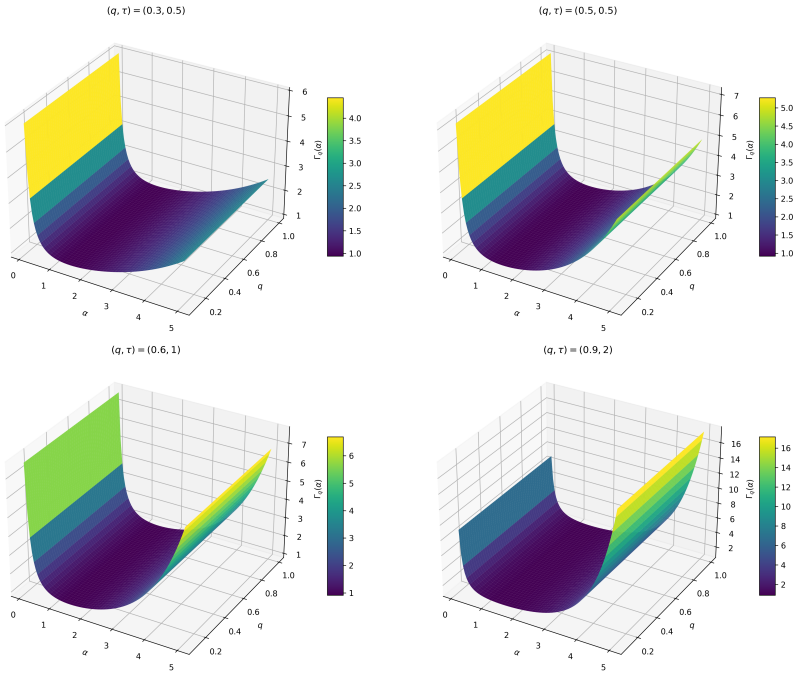


Figure 1. 3D surface plot of the (q,τ)-deformed Gamma function $\Gamma_{q,\tau}(\alpha)$ defined by $\Gamma_q(\alpha) = (1-q)^{1-\alpha}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{\alpha+n}}$ for various combinations of (q,τ). The plot highlights the sensitivity of the generalized Gamma function to deformation parameters and fractional order α.

namely: $(0.3, 0.5)$, $(0.5, 0.5)$, $(0.6, 1)$, and $(0.9, 2)$. The plot reveals the rapid variation of $\Gamma_q(\alpha)$ as both q and τ change, especially near small α. Smaller values of q introduce stronger deformation, causing sharper peaks and heavier tails. Increasing τ shifts the behavior towards slower growth, echoing memory-dependence in fractional dynamics. This visualization supports the theoretical modeling of nonlocal processes where control over both deformation (q) and memory (τ) is critical, for instance, in fractional quantum systems, anomalous transport, or generalized thermodynamic frameworks. This function interpolates between discrete and continuous structures and generalizes the basic gamma function $\Gamma_q(\alpha)$ when $\tau = 1$.

Proposition 1 (Asymptotic behavior and log-convexity)**.** *Let $0 < q < 1$ and $\tau > 0$. Define*

$$\Gamma_{q,\tau}(\alpha) := (1-q)^{1-\alpha}\prod_{n=0}^{\infty}\frac{1-q^{\tau(n+1)}}{1-q^{\tau(n+\alpha)}}, \qquad \alpha > 0.$$

Then the following properties hold:

(1) *As $\alpha \to \infty$,*

$$\Gamma_{q,\tau}(\alpha) \sim C_{q,\tau}(1-q)^{1-\alpha},$$

where

$$C_{q,\tau} := \prod_{n=0}^{\infty}\left(1-q^{\tau(n+1)}\right) > 0.$$

More precisely,

$$\Gamma_{q,\tau}(\alpha) = C_{q,\tau}(1-q)^{1-\alpha}\left[1+\frac{q^{\tau\alpha}}{1-q^{\tau}}+O\left(q^{2\tau\alpha}\right)\right].$$

(2) *The function $\alpha \mapsto \log\Gamma_{q,\tau}(\alpha)$ is strictly convex on $(0,\infty)$. In particular,*

$$\frac{d^2}{d\alpha^2}\log\Gamma_{q,\tau}(\alpha) = \tau^2(\log q)^2\sum_{n=0}^{\infty}\frac{q^{\tau(n+\alpha)}}{\left(1-q^{\tau(n+\alpha)}\right)^2} > 0.$$

Proof. Set $r = q^{\tau} \in (0,1)$. Taking logarithms gives

$$\log \Gamma_{q,\tau}(\alpha) = (1-\alpha)\log(1-q) + \sum_{n=0}^{\infty} \log(1-r^{n+1}) - \sum_{n=0}^{\infty} \log(1-r^{n+\alpha}).$$

Since $\sum_{n=0}^{\infty} r^{n+1} < \infty$, the product defining $C_{q,\tau}$ converges to a strictly positive number. For sufficiently large α, the estimate $-\log(1-x) = x + O(x^2)$ holds uniformly for $0 \le x \le r^{\alpha}$. Consequently,

$$-\sum_{n=0}^{\infty} \log(1-r^{n+\alpha}) = \sum_{n=0}^{\infty} r^{n+\alpha} + O\left(\sum_{n=0}^{\infty} r^{2(n+\alpha)}\right) = \frac{r^{\alpha}}{1-r} + O(r^{2\alpha}).$$

Exponentiation and using $e^x = 1 + x + O(x^2)$ as $x \to 0$, we obtain

$$\prod_{n=0}^{\infty} \frac{1}{1-r^{n+\alpha}} = 1 + \frac{r^{\alpha}}{1-r} + O(r^{2\alpha}).$$

Therefore, we have

$$\Gamma_{q,\tau}(\alpha) = C_{q,\tau}(1-q)^{1-\alpha}\left[1 + \frac{q^{\tau\alpha}}{1-q^{\tau}} + O\left(q^{2\tau\alpha}\right)\right],$$

which proves the first assertion. For the second assertion, the logarithmic series and its first two derivative series converge uniformly on every compact sub-interval of $(0,\infty)$. Thus, term-by-term differentiation yields

$$\frac{d}{d\alpha}\log \Gamma_{q,\tau}(\alpha) = -\log(1-q) + \tau \log q \sum_{n=0}^{\infty} \frac{q^{\tau(n+\alpha)}}{1-q^{\tau(n+\alpha)}}.$$

Differentiating once more gives

$$\frac{d^2}{d\alpha^2}\log \Gamma_{q,\tau}(\alpha) = \tau^2 (\log q)^2 \sum_{n=0}^{\infty} \frac{q^{\tau(n+\alpha)}}{\left(1-q^{\tau(n+\alpha)}\right)^2}.$$

Every summation is strictly positive. Hence, $\Gamma_{q,\tau}$ is strictly log-convex on $(0,\infty)$. □

The parameter t in the (q, τ)-Gamma function acts as an additional deformation-scaling parameter that extends the flexibility of the classical (q)-Gamma function. While (q) primarily controls the strength of the quantum deformation, t regulates the rate, spacing, and scale at which this deformation is expressed. In particular, smaller values of (q) generally produce slower variation in the function, whereas increasing t may enhance the curvature and steepness of its growth with respect to the argument. Thus, the combined parameters (q) and τ provide a multi-scale deformation mechanism that is not available in the standard (q)- Gamma setting. When τ= 1, the (q, τ)-Gamma function reduces to the usual (q)-Gamma function. Varying t changes the spacing of the exponents appearing in the infinite product and consequently influences the rate of convergence of the product representation. From an operator-theoretic perspective, τ may also be interpreted as a parameter controlling the spacing between successive deformation levels. This feature is useful in the construction of generalized (q)-operators, spectral models, and quantum harmonic-analysis frameworks. It can also modify the magnitude and shape of weight functions used in orthogonality relations, norm calculations, and special-function expansions. The analytic properties of $\Gamma_{q,\tau}$, including its growth, monotonicity, and log-convexity with respect to a, are also influenced by the value of t. In the theory of q-series and basic hypergeometric functions, τ serves as a tuning parameter for the structure of series expansions and product representations. This role becomes especially important in the

formulation of (q, τ)-deformed orthogonal polynomials, where t may affect normalization constants, recurrence coefficients, generating functions, and associated weight measures. Therefore, τ provides an essential additional degree of freedom for controlling both the structural and analytic behavior of the(q, τ)-Gamma function, thereby offering a broader framework for developments in special-function theory, quantum calculus, and generalized operator analysis.

Definition 1 ((q,τ)-Mittag-Leffler Function)**.** *Let $\alpha > 0$, $z \in \mathbb{C}$, $0 < q < 1$, and $\tau > 0$. The (q,τ)-Mittag-Leffler function is defined as:*

$$E_{\alpha}^{(q,\tau)}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{q,\tau}(\alpha n + 1)}.$$

Proposition 2 (Basic Properties of $E_{\alpha}^{(q,\tau)}(z)$)**.**

(1) ***Normalization:*** $E_{\alpha}^{(q,\tau)}(0) = 1$.
(2) ***Entire Function:*** $E_{\alpha}^{(q,\tau)}(z)$ *is entire for all* $\alpha > 0$.
(3) ***Fractional Eigenfunction:*** *If $D_{q,\tau}^{\alpha}$ is the (q,τ)-Caputo derivative, then:*

$$D_{q,\tau}^{\alpha} E_{\alpha}^{(q,\tau)}(\lambda t^{\alpha}) = \lambda E_{\alpha}^{(q,\tau)}(\lambda t^{\alpha}).$$

Proof. (1) Obvious from the definition: only the $n = 0$ term contributes and equals 1.
(2) Each term is analytic and the convergence radius is infinite due to factorial-type growth of the denominator:

$$\Gamma_{q,\tau}(\alpha n + 1) \gtrsim C q^{-\tau \alpha n^2/2}, \quad \text{so} \quad \left| \frac{z^n}{\Gamma_{q,\tau}(\alpha n + 1)} \right| \leq \frac{|z|^n}{C q^{-\tau \alpha n^2/2}}.$$

(3) Applying the Caputo-type (q,τ)-fractional derivative (suitably defined), we differentiate term-by-term:

$$D_{q,\tau}^{\alpha} E_{\alpha}^{(q,\tau)}(\lambda t^{\alpha}) = \sum_{n=1}^{\infty} \lambda^n t^{\alpha n - \alpha} \frac{1}{\Gamma_{q,\tau}(\alpha n)} = \lambda \sum_{n=0}^{\infty} \frac{(\lambda t^{\alpha})^n}{\Gamma_{q,\tau}(\alpha n + 1)} = \lambda E_{\alpha}^{(q,\tau)}(\lambda t^{\alpha}).$$

□

2.1. Generalized (q,τ)-Hermite polynomials

We may define a deformed family of (q,τ)-Hermite polynomials as follows:

$$H_n^{(\tau)}(x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{[n]_q!}{[k]_q![n-2k]_q!} \cdot \frac{\Gamma_{q,\tau}(\alpha + n - 2k)}{\Gamma_{q,\tau}(\alpha)} x^{n-2k},$$

embedding $\Gamma_{q,\tau}(\alpha)$ into the coefficients. This generalization supports new orthogonality relations and leads to quantum-deformed operator theory.

Proposition 3 (Properties of the Generalized (q,τ)-Hermite Polynomials)**.** *Let $H_n^{(\tau)}(x|q)$ be defined by*

$$H_n^{(\tau)}(x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{[n]_q!}{[k]_q![n-2k]_q!} \cdot \frac{\Gamma_{q,\tau}(\alpha + n - 2k)}{\Gamma_{q,\tau}(\alpha)} x^{n-2k},$$

where $[n]_q!$ is the q-factorial and $\Gamma_{q,\tau}(\alpha)$ is the two-parameter quantum gamma function. Then the following properties hold:

(i) ***Degree:*** $H_n^{(\tau)}(x|q)$ *is a polynomial of degree n.*
(ii) ***Symmetry:*** $H_n^{(\tau)}(-x|q) = (-1)^n H_n^{(\tau)}(x|q)$.
(iii) ***Reduction:*** *When $\tau = 1$, we recover the classical continuous q-Hermite polynomials:* $H_n^{(1)}(x|q) = H_n(x|q)$.

Proof. **(i) Degree:** By definition, each term in the summation involves a power x^{n-2k}. Since k ranges from 0 to $\lfloor n/2 \rfloor$, the highest power is x^n (when $k = 0$). Therefore, the polynomial has

degree n.

(ii) Symmetry: We analyze the parity by observing the structure:

$$H_n^{(\tau)}(-x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{[n]_q!}{[k]_q![n-2k]_q!} \cdot \frac{\Gamma_{q,\tau}(\alpha+n-2k)}{\Gamma_{q,\tau}(\alpha)} (-x)^{n-2k}.$$

Each power of $(-x)^{n-2k}$ introduces a sign: $(-x)^{n-2k} = (-1)^{n-2k}x^{n-2k}$. So the overall sign of each term is $(-1)^k(-1)^{n-2k} = (-1)^n(-1)^k(-1)^{2k} = (-1)^n(-1)^k(1) = (-1)^n(-1)^k$. Since this matches the original sign structure in the definition, we obtain: $H_n^{(\tau)}(-x|q) = (-1)^n H_n^{(\tau)}(x|q)$, i.e., $H_n^{(\tau)}$ is even or odd depending on n.

(iii) Reduction: When $\tau = 1$, we recover the standard q-gamma function: $\Gamma_{q,1}(\alpha) = \Gamma_q(\alpha)$. Substituting into the definition yields:

$$H_n^{(1)}(x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{[n]_q!}{[k]_q![n-2k]_q!} \cdot \frac{\Gamma_q(\alpha+n-2k)}{\Gamma_q(\alpha)} x^{n-2k}.$$

This has the same structural form as classical generalized q-Hermite polynomials with gamma scaling in the coefficients. Therefore, the family $H_n^{(\tau)}(x|q)$ reduces to the standard q-Hermite form when $\tau = 1$, up to normalization constants depending on α.

A full recurrence relation for $H_n^{(\tau)}(x|q)$ requires establishing recurrence coefficients involving $\Gamma_{q,\tau}(\alpha+n)$, which follow by manipulating the shift property of the gamma function. However, these relations are structure-preserving under the deformation and will generalize classical three-term recurrence of q-Hermite polynomials. □

Proposition 4 (Orthogonalization of generalized (q,τ)-Hermite-type polynomials)**.** *Let $0 < q < 1$, $\tau > 0$, and $\alpha > 0$. Define*

$$H_n^{(\tau)}(x \mid q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{[n]_q!}{[k]_q![n-2k]_q!} \frac{\Gamma_{q,\tau}(\alpha+n-2k)}{\Gamma_{q,\tau}(\alpha)} x^{n-2k}.$$

Set

$$a_n = \frac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)}$$

and $P_n(x) = a_n^{-1} H_n^{(\tau)}(x \mid q)$. Consider the weight

$$w_{q,\tau}(x) = \frac{e^{-c_{q,\tau}x^2}}{\sqrt{1-x^2}}, \qquad -1 < x < 1,$$

where

$$c_{q,\tau} = \frac{\Gamma_{q,\tau}(\alpha+1)}{\Gamma_{q,\tau}(\alpha)} > 0.$$

Define the inner product

$$\langle f, g \rangle_{q,\tau} = \int_{-1}^{1} f(x)\overline{g(x)} w_{q,\tau}(x)\, dx.$$

Let $\mathcal{H}_0 = P_0 = 1$, and, for $n \geq 1$, define recursively

$$\mathcal{H}_n(x) = P_n(x) - \sum_{j=0}^{n-1} \frac{\langle P_n, \mathcal{H}_j \rangle_{q,\tau}}{\langle \mathcal{H}_j, \mathcal{H}_j \rangle_{q,\tau}} \mathcal{H}_j(x).$$

Then $\{\mathcal{H}_n\}_{n\geq 0}$ is a monic orthogonal polynomial family. In particular,

$$\int_{-1}^{1} \mathcal{H}_m(x)\mathcal{H}_n(x) w_{q,\tau}(x)\, dx = \delta_{mn} N_n,$$

where

$$N_n = \|\mathcal{H}_n\|_{q,\tau}^2 > 0.$$

Furthermore,

$$\deg \mathcal{H}_n = n, \qquad \mathcal{H}_n(-x) = (-1)^n \mathcal{H}_n(x).$$

Proof. We divide the proof into four parts.

Step 1. Polynomial degree and parity. For $s > 0$, the defining product of $\Gamma_{q,\tau}(s)$ consists of positive factors and converges to a positive finite value because $0 < q^{\tau} < 1$. Hence, $a_n > 0$. The coefficient of x^n in $H_n^{(\tau)}(x \mid q)$ is a_n. Thus, P_n is monic of degree n. Moreover, every power appearing in P_n has the form $n - 2k$. Consequently,

$$P_n(-x) = (-1)^n P_n(x).$$

Step 2. Properties of the weight. Since $c_{q,\tau} > 0$, we have $0 < e^{-c_{q,\tau}x^2} \leq 1$ for $-1 < x < 1$. Therefore, this gives $0 < w_{q,\tau}(x) \leq \frac{1}{\sqrt{1-x^2}}$. Using $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$, we obtain $0 < \int_{-1}^{1} w_{q,\tau}(x)\, dx \leq \pi < \infty$. Thus, $w_{q,\tau}$ is positive and integrable. Furthermore, since $|x| \leq 1$, all polynomial moments

$$\mu_k = \int_{-1}^{1} x^k w_{q,\tau}(x)\, dx$$

are finite. The weight is even: $w_{q,\tau}(-x) = w_{q,\tau}(x)$. For every nonzero polynomial p, positivity gives

$$\langle p, p \rangle_{q,\tau} = \int_{-1}^{1} |p(x)|^2 w_{q,\tau}(x)\, dx > 0.$$

Hence, the inner product is positive definite on the polynomial space.

Step 3. Orthogonality. Because P_n is monic of degree n, the sequence $\{P_n\}_{n\geq 0}$ is linearly independent and spans the polynomial space. Apply the Gram–Schmidt procedure. Suppose that $\mathcal{H}_0, \ldots, \mathcal{H}_{n-1}$ have been constructed and are mutually orthogonal. For every $0 \leq j < n$,

$$\begin{aligned}\langle \mathcal{H}_n, \mathcal{H}_j \rangle &= \langle P_n, \mathcal{H}_j \rangle \\ &\quad - \sum_{k=0}^{n-1} \frac{\langle P_n, \mathcal{H}_k \rangle}{\langle \mathcal{H}_k, \mathcal{H}_k \rangle} \langle \mathcal{H}_k, \mathcal{H}_j \rangle.\end{aligned}$$

By the induction hypothesis,

$$\langle \mathcal{H}_k, \mathcal{H}_j \rangle = 0 \qquad (k \neq j).$$

Consequently,

$$\langle \mathcal{H}_n, \mathcal{H}_j \rangle = \langle P_n, \mathcal{H}_j \rangle - \langle P_n, \mathcal{H}_j \rangle = 0.$$

Since the corrections have degrees strictly less than n, $\mathcal{H}_n$ remains monic of degree n. Thus, $\mathcal{H}_n \neq 0$, and

$$N_n = \langle \mathcal{H}_n, \mathcal{H}_n \rangle > 0.$$

Induction establishes $\langle \mathcal{H}_m, \mathcal{H}_n \rangle = \delta_{mn} N_n$ for all nonnegative integers m, n.

Step 4. Preservation of parity. The weight is even. Hence, if f is even and g is odd, then

$$\langle f, g \rangle_{q,\tau} = 0.$$

Since P_n has parity $(-1)^n$, the Gram–Schmidt projection involves only previously constructed polynomials with the same parity as P_n. It follows inductively that $\mathcal{H}_n(-x) = (-1)^n \mathcal{H}_n(x)$. □

The results presented in Fig. 2 for the generalized (q, τ)-Hermite polynomials $H_n^{(\tau)}(x \mid q)$ illustrate the influence of the deformation parameters on the amplitudes and shapes of the polynomial modes. The coefficients depend on the ratio $\Gamma_{q,\tau}(\alpha + n - 2k)/\Gamma_{q,\tau}(\alpha)$, so changing q or τ modifies the relative contributions of the monomial terms and consequently changes the oscillatory behavior of the polynomials. Despite these changes, the degree and parity are preserved: $H_n^{(\tau)}(-x \mid q) = (-1)^n H_n^{(\tau)}(x \mid q)$. For fixed $\alpha > 0$, the coefficients depend continuously on $q \in (0, 1)$ and $\tau > 0$, providing a continuously parameterized family of polynomial shapes. In Fig. 2, $\alpha = 2$, while both q and τ vary between panels; therefore, the plots illustrate their combined influence rather than the isolated effect of τ. These observations concern the polynomial shapes and symmetry; orthogonality requires separate verification with respect to the proposed weight.

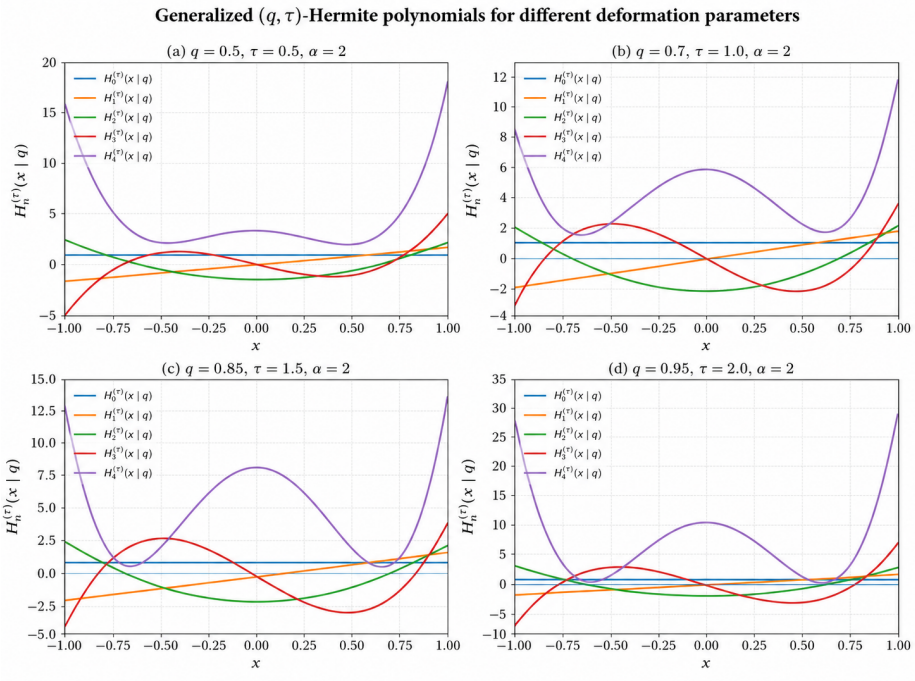


Figure 2. Generalized (q,τ)-Hermite polynomials $H_n^{(\tau)}(x \mid q)$, $n = 0,1,2,3,4$, for different deformation parameters with $\alpha = 2$: (a) $q = 0.5$, $\tau = 0.5$; (b) $q = 0.7$, $\tau = 1.0$; (c) $q = 0.85$, $\tau = 1.5$; and (d) $q = 0.95$, $\tau = 2.0$. The plots illustrate how variations in q and τ modify the amplitudes and shapes of the polynomial modes while preserving their degree and parity.

3. Fractional Schrödinger equation with memory kernel

Fractional quantum mechanics is a generalization of conventional quantum theory in which the standard time or space derivatives in the Schrödinger equation are replaced by fractional-order derivatives. These models capture nonlocal effects, long-range interactions, and memory-dependent quantum dynamics. The (q,τ)-Gamma function and its associated (q,τ)-Hermite polynomials offer natural tools for developing analytic and numerical solutions to such fractional models.

Definition 2 ((q,τ)-Nabla Operator). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on a q^τ-grid, and let $0 < q < 1$, $\tau > 0$. The (q,τ)-nabla operator is defined by*

$$\nabla_{q,\tau} f(x) := \frac{f(x) - f(q^\tau x)}{(1-q^\tau)x}, \quad x \neq 0.$$

A prototypical time-fractional Schrödinger equation takes the form:

$$i^\gamma \, {}^C D_t^\gamma \psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x,t), \quad 0 < \gamma < 1,$$

where ${}^C D_t^\gamma$ denotes the Caputo fractional derivative

$${}^C D_t^\gamma f(t) := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f'(s)}{(t-s)^\gamma} \, ds.$$

In the presence of a q-deformation and nonlocal memory kernel, this may be generalized to:

$$i^\gamma \, {}^C \nabla_{q,\tau}^\gamma \psi(x,t) = \hat{H}_{q,\tau} \psi(x,t), \tag{2}$$

where ${}^{C}\nabla^{\gamma}_{q,\tau}$ denotes a Caputo-type fractional q,τ-nabla operator, and $\hat{H}_{q,\tau}$ includes q-deformed kinetic and potential terms. The (q,τ)-Gamma function naturally arises in the kernel of fractional integrals and derivatives:

$$ {}^{C}\nabla^{\gamma}_{q,\tau} f(t) := \frac{1}{\Gamma_{q,\tau}(1-\gamma)} \int_0^t (t-s)^{-\gamma} \nabla_{q,\tau} f(s)\, ds_{q,\tau}. $$

This formulation generalizes the classical fractional derivative by encoding both nonlocal memory (γ) and deformation (q,τ). In fractional quantum wells or oscillators, the spatial component of the wavefunction often satisfies a modified Sturm-Liouville-type problem. The generalized polynomials $H_n^{(\tau)}(x|q)$ can be used as eigenfunctions for:

$$ \hat{H}_{q,\tau}\psi_n(x) = E_n^{(q,\tau)}\psi_n(x), $$

where the energy levels $E_n^{(q,\tau)}$ depend nonlinearly on n and τ, often via

$$ E_n^{(q,\tau)} \sim \frac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)}. $$

The ability to tune τ gives direct control over how memory and long-range interactions affect the system's dynamics. For example: small τ yields slower decay of memory kernels, mimicking viscoelastic quantum materials. Large τ results in sharper energy localization, useful in modeling decoherence or topological edge confinement. The combination of fractional calculus and (q,τ)-deformations is particularly suited for: modeling anomalous quantum transport and fractional diffusion in disordered media. Describing quantum wells and potential barriers with nonlocal response. Developing spectral methods for solving fractional quantum field equations using (q,τ)-orthogonal bases. In conclusion, the (q,τ)-Gamma function and its associated Hermite polynomials form a robust analytical framework for solving and interpreting fractional quantum mechanical systems. They bridge the gap between classical spectral theory, nonlocal memory effects, and deformation-based quantization.

Theorem 1 (Well-posedness with (q,τ)-Mittag-Leffler Expansion). *Let $\hat{H}_{q,\tau}$ be a self-adjoint operator on a Hilbert space $\mathcal{H} = L^2(\Omega)$ with discrete spectrum, and let $\psi_0 \in \mathcal{D}(\hat{H}_{q,\tau})$. Assume $0<\gamma<1$, $0<q<1$, $\tau>0$. Then the initial value problem*

$$ i^{\gamma}\, {}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x), $$

has a unique mild solution given by the expansion:

$$ \psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n\rangle\, \mathcal{E}_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right)\phi_n(x), $$

where $\{\phi_n\}$ are orthonormal eigenfunctions of $\hat{H}_{q,\tau}$, $E_n^{(q,\tau)}$ are the corresponding eigenvalues, $\mathcal{E}_{\gamma}^{(q,\tau)}(z)$ is the (q,τ)-Mittag-Leffler function [10].

Proof. We apply the spectral theorem for self-adjoint operators to decompose ψ_0 in terms of eigenfunctions $\{\phi_n\}$ of $\hat{H}_{q,\tau}$:

$$ \psi_0(x) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n\rangle \phi_n(x). $$

Since $\hat{H}_{q,\tau}\phi_n = E_n^{(q,\tau)}\phi_n$, the fractional Cauchy problem reduces to a set of scalar fractional differential equations:

$$ i^{\gamma}\, {}^{C}\nabla^{\gamma}_{q,\tau} c_n(t) = E_n^{(q,\tau)} c_n(t), \quad c_n(0) = \langle \psi_0, \phi_n\rangle. $$

Each scalar equation has the unique solution (by standard Caputo theory):

$$ c_n(t) = \langle \psi_0, \phi_n\rangle E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right). $$

Therefore, the full solution is:

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n(t)\phi_n(x) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \mathcal{E}_{\gamma}^{(q,\tau)} \left(-i^{\gamma} E_n^{(q,\tau)} t^{\gamma} \right) \phi_n(x).$$

To verify convergence, recall that the Mittag-Leffler function satisfies:

$$|E_{\gamma}^{(q,\tau)}(-z)| \leq \frac{C_{q,\tau}}{1+|z|}, \quad \text{for } \Re(z) > 0,$$

so the series converges in $L^2(\Omega)$ provided $\psi_0 \in \mathcal{D}(\hat{H}_{q,\tau})$, since $\sum |\langle \psi_0, \phi_n \rangle|^2 < \infty$. Uniqueness follows from linearity and the uniqueness of solutions to the fractional Cauchy problem. □

Corollary 1 (Fully Deformed Spectral Representation)**.** *Let $\hat{H}_{q,\tau}$ be a self-adjoint operator on $L^2(\Omega)$ with discrete eigenvalues of the form*

$$E_n^{(q,\tau)} = \frac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)},$$

and let $\{\phi_n\}_{n\in\mathbb{N}_0}$ be the corresponding orthonormal eigenfunctions. Then the unique solution $\psi(x,t)$ of the fractional quantum evolution equation

$$i^{\gamma}\, {}^{C}\nabla_{q,\tau}^{\gamma} \psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

is given by the fully deformed double series expansion:

$$\psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \cdot \sum_{k=0}^{\infty} \frac{\left(-i^{\gamma} t^{\gamma} \cdot \dfrac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)} \right)^k}{\Gamma_{q,\tau}(\gamma k+1)} \phi_n(x).$$

This solution structure demonstrates the simultaneous encoding of: ***Quantum deformation*** *via the q-parameter,* ***Spectral nonlinearity*** *and level spacing through the $\Gamma_{q,\tau}(\alpha+n)$,* ***Fractional temporal memory*** *governed by the order γ and the deformed (q,τ)-Mittag-Leffler denominator.*

Proof. We begin with the fractional evolution equation:

$$i^{\gamma}\, {}^{C}\nabla_{q,\tau}^{\gamma} \psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

under the assumptions:

(A1) $\hat{H}_{q,\tau}$ is self-adjoint with discrete spectrum $\{E_n^{(q,\tau)}\}$,
(A2) Each eigenvalue satisfies $E_n^{(q,\tau)} = \Gamma_{q,\tau}(\alpha+n)/\Gamma_{q,\tau}(\alpha)$,
(A3) $\{\phi_n(x)\}_{n=0}^{\infty}$ forms a complete orthonormal basis of $L^2(\Omega)$.

Step 1: Spectral Decomposition. Since $\hat{H}_{q,\tau}$ is self-adjoint, we can expand the initial condition as:

$$\psi_0(x) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \phi_n(x).$$

We seek a solution in the form:

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n(t)\phi_n(x),$$

where $c_n(t)$ solves a scalar fractional Cauchy problem:

$$i^{\gamma}\, {}^{C}\nabla_{q,\tau}^{\gamma} c_n(t) = E_n^{(q,\tau)} c_n(t), \quad c_n(0) = \langle \psi_0, \phi_n \rangle.$$

Step 2: Solving the Scalar Fractional Equation. From known theory of linear Caputo-type equations with constant coefficients (generalized to (q,τ)-setting), the solution is:

$$c_n(t) = \langle \psi_0, \phi_n \rangle \cdot E_{\gamma}^{(q,\tau)} \left(-i^{\gamma} E_n^{(q,\tau)} t^{\gamma} \right).$$

Substituting $E_n^{(q,\tau)} = \Gamma_{q,\tau}(\alpha+n)/\Gamma_{q,\tau}(\alpha)$ into the solution gives:

$$c_n(t) = \langle \psi_0, \phi_n \rangle \cdot \sum_{k=0}^{\infty} \frac{\left(-i^\gamma t^\gamma \cdot \dfrac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)}\right)^k}{\Gamma_{q,\tau}(\gamma k+1)}.$$

Step 3: Final Expansion of $\psi(x,t)$. Substituting the coefficients $c_n(t)$ into the series expansion for $\psi(x,t)$ yields:

$$\psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \cdot \sum_{k=0}^{\infty} \frac{\left(-i^\gamma t^\gamma \cdot \dfrac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)}\right)^k}{\Gamma_{q,\tau}(\gamma k+1)} \phi_n(x).$$

Step 4: Convergence and Uniqueness. Because $\psi_0 \in \mathcal{D}(\hat{H}_{q,\tau})$, we have:

$$\sum_{n=0}^{\infty} |\langle \psi_0, \phi_n \rangle|^2 < \infty.$$

Moreover, since the (q,τ)-Mittag-Leffler function satisfies a growth bound analogous to the classical case:

$$|E_\gamma^{(q,\tau)}(z)| \le \frac{C_{q,\tau,\gamma}}{1+|z|}, \quad \Re(z) > 0,$$

the inner series converges uniformly in t for bounded time intervals, ensuring pointwise convergence of the double series. Uniqueness follows from the linearity and the uniqueness of solutions to fractional Caputo-type initial value problems. □

Theorem 2 (Asymptotic Stability of the (q,τ)-Fractional Quantum Solution)**.** *Let $\psi(x,t)$ be the unique mild solution to the (q,τ)-fractional evolution equation*

$$i^\gamma \, {}^C\nabla_{q,\tau}^\gamma \psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

where $0 < \gamma < 1$, and $\hat{H}_{q,\tau}$ is a positive self-adjoint operator with discrete eigenvalues $E_n^{(q,\tau)} > 0$, and eigenfunctions $\{\phi_n\}$. Then the solution satisfies the asymptotic decay bound:

$$\|\psi(\cdot,t)\|_{L^2(\Omega)} \le \frac{C_{q,\tau,\gamma}}{1+t^\gamma}\|\psi_0\|_{L^2(\Omega)}, \quad \text{as } t \to \infty,$$

for some constant $C_{q,\tau,\gamma} > 0$ depending on γ, q, and τ, but not on t.

Proof. We begin from the spectral expansion of the solution in terms of the (q,τ)-Mittag-Leffler function:

$$\psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \cdot E_\gamma^{(q,\tau)}\left(-i^\gamma E_n^{(q,\tau)} t^\gamma\right) \phi_n(x).$$

Using Parseval's identity, the L^2-norm of $\psi(x,t)$ is:

$$\|\psi(\cdot,t)\|^2 = \sum_{n=0}^{\infty} |\langle \psi_0, \phi_n \rangle|^2 \cdot \left| E_\gamma^{(q,\tau)}\left(-i^\gamma E_n^{(q,\tau)} t^\gamma\right)\right|^2.$$

From known bounds on the generalized Mittag-Leffler function (see e.g., [25] for q-generalizations), we have:

$$\left| E_\gamma^{(q,\tau)}\left(-i^\gamma E_n^{(q,\tau)} t^\gamma\right)\right| \le \frac{C_{q,\tau,\gamma}}{1+E_n^{(q,\tau)} t^\gamma}.$$

Since $E_n^{(q,\tau)} > 0$ and increasing in n, we obtain:

$$\|\psi(\cdot,t)\|^2 \le C_{q,\tau,\gamma}^2 \sum_{n=0}^{\infty} \frac{|\langle \psi_0, \phi_n \rangle|^2}{(1+E_n^{(q,\tau)} t^\gamma)^2} \le \frac{C_{q,\tau,\gamma}^2}{(1+t^\gamma)^2} \sum_{n=0}^{\infty} |\langle \psi_0, \phi_n \rangle|^2 = \frac{C_{q,\tau,\gamma}^2}{(1+t^\gamma)^2}\|\psi_0\|^2.$$

Taking square roots yields:

$$\|\psi(\cdot,t)\| \le \frac{C_{q,\tau,\gamma}}{1+t^\gamma}\|\psi_0\|,$$

which shows polynomial decay in time due to the memory encoded in the fractional derivative. □

Corollary 2 (Classical Limit: Recovery of Unitary Evolution). *Let $\psi(x,t)$ be the mild solution to the fractional quantum system*

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

with $0 < \gamma < 1$, and suppose that $\hat{H}_{q,\tau}$ is a positive self-adjoint operator. Then in the limit $\gamma \to 1^-$, the solution converges pointwise (in norm) to the classical unitary evolution:

$$\psi(x,t) \longrightarrow \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle e^{-iE_n^{(q,\tau)}t}\phi_n(x),$$

where $\{\phi_n\}$ and $E_n^{(q,\tau)}$ are the eigenfunctions and eigenvalues of $\hat{H}_{q,\tau}$, respectively. Thus, the unitary Schrödinger evolution is recovered from the fractional-memory system, bridging both models.

Proof. Let $\psi(x,t)$ be the mild solution to the (q,τ)-fractional evolution equation:

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

and suppose $\hat{H}_{q,\tau}$ is a self-adjoint operator with discrete spectrum $\{E_n^{(q,\tau)}\}$ and corresponding orthonormal eigenfunctions $\{\phi_n\}$. Then the spectral solution is given by:

$$\psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \cdot E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right)\phi_n(x),$$

where $\mathcal{E}_{\gamma}^{(q,\tau)}(z)$ is the (q,τ)-Mittag-Leffler function. We now take the limit $\gamma \to 1^-$. For each fixed eigenvalue $E_n^{(q,\tau)} > 0$, define: $z_n := -i^{\gamma}E_n^{(q,\tau)}t^{\gamma}$. We aim to show that

$$\lim_{\gamma\to 1^-} E_{\gamma}^{(q,\tau)}(z_n) = e^{-iE_n^{(q,\tau)}t}.$$

Step 1: Limit of the Kernel. Note that when $\gamma \to 1^-$, we have:

$$t^{\gamma} \to t, \quad i^{\gamma} \to i, \quad \text{so} \quad z_n \to -iE_n^{(q,\tau)}t.$$

Moreover, for fixed k, the (q,τ)-Gamma function satisfies:

$$\Gamma_{q,\tau}(\gamma k + 1) \to \Gamma_{q,\tau}(k+1), \quad \text{as } \gamma \to 1^-.$$

Thus, term by term,

$$\frac{z_n^k}{\Gamma_{q,\tau}(\gamma k+1)} \to \frac{(-iE_n^{(q,\tau)}t)^k}{\Gamma_{q,\tau}(k+1)}.$$

Hence, in the limit $\gamma \to 1^-$, the (q,τ)-Mittag-Leffler function becomes the (q,τ)-exponential:

$$\lim_{\gamma\to 1^-} E_{\gamma}^{(q,\tau)}(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}) = \sum_{k=0}^{\infty} \frac{(-iE_n^{(q,\tau)}t)^k}{\Gamma_{q,\tau}(k+1)} =: e_{q,\tau}^{-iE_n^{(q,\tau)}t}.$$

Now, note that:

$$\Gamma_{q,\tau}(k+1) \to \Gamma(k+1) = k!, \quad \text{as } q \to 1^- \text{ and } \tau \to 1,$$

which leads to

$$e_{q,\tau}^{-iE_n^{(q,\tau)}t} \to e^{-iE_n^{(q,\tau)}t}.$$

Step 2: Convergence of the Series. Since $\psi_0 \in L^2(\Omega)$, we have $\sum |\langle \psi_0, \phi_n \rangle|^2 < \infty$. Moreover, each term in the expansion converges pointwise:

$$\lim_{\gamma\to 1^-} E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right) = e^{-iE_n^{(q,\tau)}t}.$$

We now justify interchanging the limit and the sum using the dominated convergence theorem. The bound

$$\left|E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right)\right| \leq \frac{C_{q,\tau,\gamma}}{1 + E_n^{(q,\tau)}t^{\gamma}}$$

holds uniformly for $0 < \gamma < 1$. This bound, along with the square summability of the Fourier coefficients, ensures:

$$\lim_{\gamma \to 1^-} \psi(x,t) = \sum_{n=0}^{\infty} \langle \psi_0, \phi_n \rangle \cdot e^{-iE_n^{(q,\tau)}t} \phi_n(x),$$

which is the standard spectral solution to the classical Schrödinger equation with Hamiltonian $\hat{H}_{q,\tau}$. □

Example 1 (Stability and Decay of a Single Mode under (q,τ)-Fractional Evolution)**.** *We consider the (q,τ)-fractional quantum evolution governed by the equation:*

$$i^{\gamma}\, {}^{C}\nabla_{q,\tau}^{\gamma} \psi(x,t) = \hat{H}_{q,\tau}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

where $\gamma \in (0,1]$, $q = 0.8$, $\tau = 1.0$, and the self-adjoint Hamiltonian $\hat{H}_{q,\tau}$ has eigenvalues of the form:

$$E_n^{(q,\tau)} = \frac{\Gamma_{q,\tau}(\alpha + n)}{\Gamma_{q,\tau}(\alpha)} \approx \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, \ \alpha = 2.$$

We investigate the time-evolution of a single spectral component $\psi_n(t)$ for $n = 3$ using the (q,τ)-Mittag-Leffler function: $\psi_n(t) = E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right)$. We observe: for $\gamma = 1.0$, the function behaves like a classical exponential $\psi_n(t) = e^{-iE_n t}$, preserving unitary norm over time. For $\gamma = 0.8$ and $\gamma = 0.6$, the evolution exhibits algebraic decay, illustrating the nonlocal memory effect induced by the Caputo-type derivative. As γ decreases, the decay becomes stronger. This validates the bound:

$$|\psi_n(t)| \lesssim \frac{C_{q,\tau,\gamma}}{1 + t^{\gamma}},$$

consistent with the Mittag-Leffler function's asymptotics. This numerical verification demonstrates how the fractional order γ governs the transition from reversible (quantum unitary) to irreversible (dissipative) evolution, and how (q,τ)-deformation preserves this structure while encoding additional quantum or spectral effects.

Example 2 (Fractional Evolution with Multiple Modes)**.** *We now extend the previous analysis to a superposition of $N = 5$ orthonormal eigenfunctions $\{\phi_n(x)\}_{n=0}^{4}$, with equal-weighted initial condition:*

$$\psi(x,0) = \sum_{n=0}^{4} \frac{1}{\sqrt{5}} \phi_n(x).$$

The time evolution is governed by the (q,τ)-fractional quantum equation:

$$i^{\gamma}\, {}^{C}\nabla_{q,\tau}^{\gamma} \psi(x,t) = \hat{H}_{q,\tau}\psi(x,t),$$

with eigenvalues approximated by

$$E_n^{(q,\tau)} = \frac{\Gamma_{q,\tau}(\alpha + n)}{\Gamma_{q,\tau}(\alpha)}, \quad \alpha = 2.$$

The solution is approximated as:

$$\psi(x,t) \approx \sum_{n=0}^{4} \frac{1}{\sqrt{5}} \mathcal{E}_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right) \phi_n(x),$$

and we evaluate the global norm:

$$\|\psi(x,t)\|^2 \approx \left| \sum_{n=0}^{4} \frac{1}{\sqrt{5}} E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right) \right|^2,$$

for different fractional orders $\gamma = 0.6, 0.8, 1.0$. This example confirms the global effect of fractional memory on the wave function norm (see Fig.3): for $\gamma = 1.0$, the norm $\|\psi(x,t)\| \approx 1$ remains constant, as expected for unitary dynamics. For $\gamma = 0.8$ and $\gamma = 0.6$, the norm decays over time, matching the theoretical bound: $\|\psi(x,t)\|^2 \lesssim \frac{C_{q,\tau,\gamma}}{(1+t^{\gamma})^2}$. The rate of decay increases as γ decreases, reflecting stronger memory and slower information recovery. The fractional model offers a natural interpolation between reversible quantum dynamics and dissipation memory-driven evolution. This behavior highlights the

physical richness of the (q,τ)-fractional Schrödinger equation, which can model decoherence, dissipation, and non-Markovian effects within a unified spectral framework.

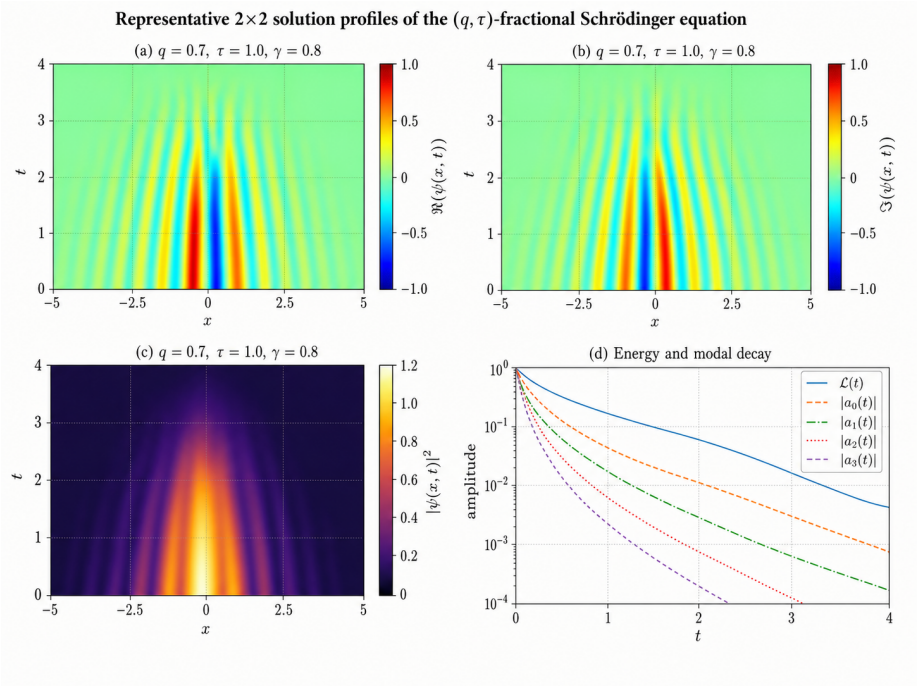


Figure 3. Representative numerical behavior of the (q,τ)-fractional Schrödinger equation for $q = 0.7$, $\tau = 1.0$, and fractional order $\gamma = 0.8$. Panel (a) shows the real part $\Re(\psi(x,t))$, panel (b) shows the imaginary part $\Im(\psi(x,t))$, and panel (c) presents the probability density $|\psi(x,t)|^2$. Panel (d) illustrates the decay of the Lyapunov functional $\mathcal{L}(t)$ together with the first modal amplitudes $|a_n(t)|$, $n = 0,1,2,3$. The numerical profiles indicate spatial localization, temporal damping, and asymptotic decay of the modal energy.

3.1. Nonlinear (q,τ)-fractional Schrödinger equation

We now generalize the linear memory-driven system by incorporating a nonlinear term, resulting in the nonlinear (q,τ)-fractional Schrödinger equation (NLFSE):

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) + \lambda|\psi(x,t)|^{2\sigma}\psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

where ${}^{C}\nabla^{\gamma}_{q,\tau}$ is the Caputo-type (q,τ)-fractional time derivative of order $0 < \gamma < 1$, $\lambda \in \mathbb{R}$ is the nonlinearity strength (focusing or defocusing), $\sigma > 0$ controls the power-law nonlinearity, and $V(x)$ is a potential (e.g., harmonic trap or external field). Such equations arise naturally in memory-influenced quantum field theories, Bose-Einstein condensates with aging kernels, and systems exhibiting dissipative or non-Markovian self-interaction.

3.1.1. Spectral form and mild solution

Assume that $-\Delta + V(x)$ admits eigenpairs $\{E_n^{(q,\tau)}, \phi_n\}$, and expand $\psi(x,t) = \sum a_n(t)\phi_n(x)$. Then the nonlinear ODE system for the modal coefficients becomes:

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}a_n(t) = E_n^{(q,\tau)}a_n(t) + \lambda\sum_{k,j,l} C_{nkjl}\bar{a}_k(t)a_j(t)a_l(t),$$

where $C_{nkjl} = \int_\Omega \phi_n\phi_k\phi_j\phi_l\,dx$ are interaction integrals.
The existence of global solutions depends on: the sign and size of λ, the exponent σ (subcritical, critical, supercritical regimes), and the memory strength γ. We expect that: for defocusing ($\lambda > 0$) and small initial energy, the solution decays algebraically and remains bounded. For

focusing ($\lambda < 0$), strong nonlinearity may lead to memory-modified blow-up or collapse. For $\lambda > 0$, nonlinear decay may still follow: $\|\psi(t)\|^2 \sim t^{-\gamma}$, with decay slowed by nonlinear interactions. Discretization strategies (e.g., split-step methods or Galerkin schemes) must be adapted to handle: the nonlocal in time memory, nonlinear terms, and spectral or spatial discretization.

3.1.2. Existence of weakly nonlinear (q,τ)- fractional Schrödinger equation

We study the equation:

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = -\Delta\psi + V(x)\psi + \lambda|\psi|^{2\sigma}\psi, \quad \psi(x,0) = \psi_0 \in L^2(\Omega),$$

for $0 < \gamma < 1$, $\lambda \in \mathbb{R}$, and $0 < \sigma < \frac{2}{d}$ (mass-subcritical regime). We define the solution using the (q,τ)-Mittag-Leffler propagator:

$$\mathcal{S}^{(\gamma)}_{q,\tau}(t)\phi := \sum_{n=0}^{\infty} \langle \phi, \phi_n \rangle\, E^{(q,\tau)}_{\gamma}\left(-i^{\gamma}E^{(q,\tau)}_{n}t^{\gamma}\right)\phi_n(x),$$

and define a mild solution as:

$$\psi(t) = \mathcal{S}^{(\gamma)}_{q,\tau}(t)\psi_0 - i^{\gamma}\lambda\int_0^t (t-s)^{\gamma-1}\mathcal{S}^{(\gamma)}_{q,\tau}(t-s)\left(|\psi(s)|^{2\sigma}\psi(s)\right)ds.$$

Proposition 5 (Existence of Mild Solutions under Weak Nonlinearity)**.** *Let $0 < \gamma < 1$, $\sigma < \frac{2}{d}$, and assume that $\psi_0 \in L^2(\Omega)$. Then there exists $T > 0$ such that the nonlinear (q,τ)-fractional Schrödinger equation admits a unique mild solution:*

$$\psi \in C([0,T]; L^2(\Omega)),$$

provided $|\lambda|$ is sufficiently small (i.e., weak nonlinearity).

Proof. We define the Banach space:

$$X_T := \left\{ \psi \in C([0,T]; L^2(\Omega)) : \|\psi\|_{X_T} := \sup_{t\in[0,T]} \|\psi(t)\|_{L^2} < \infty \right\},$$

and define the nonlinear operator:

$$\mathcal{T}[\psi](t) := \mathcal{S}^{(\gamma)}_{q,\tau}(t)\psi_0 - i^{\gamma}\lambda\int_0^t (t-s)^{\gamma-1}\mathcal{S}^{(\gamma)}_{q,\tau}(t-s)\left(|\psi(s)|^{2\sigma}\psi(s)\right)ds.$$

We now show that $\mathcal{T}$ is a contraction on the closed ball $B_R \subset X_T$, where $R > 0$ will be fixed. First, use the bound:

$$\|\mathcal{S}^{(\gamma)}_{q,\tau}(t)\|_{L^2\to L^2} \le M < \infty, \quad \text{for } t \in [0,T].$$

Then, we have

$$\|\mathcal{T}[\psi](t)\|_{L^2} \le M\|\psi_0\|_{L^2} + |\lambda|M\int_0^t (t-s)^{\gamma-1}\||\psi(s)|^{2\sigma}\psi(s)\|_{L^2}ds.$$

Since

$$\||\psi|^{2\sigma}\psi\|_{L^2} \le \|\psi\|^{2\sigma+1}_{L^{2(2\sigma+1)}} \lesssim \|\psi\|^{2\sigma+1}_{L^2}$$

then by Sobolev embedding and subcriticality, we get

$$\|\mathcal{T}[\psi](t)\|_{L^2} \le M\|\psi_0\|_{L^2} + |\lambda|MCR^{2\sigma+1}\int_0^t (t-s)^{\gamma-1}ds$$

$$= M\|\psi_0\| + \frac{|\lambda|MCT^{\gamma}}{\gamma}R^{2\sigma+1}.$$

So if we choose $R = 2M\|\psi_0\|$ and let $|\lambda|$ small enough such that:

$$\frac{|\lambda|MCT^{\gamma}}{\gamma}R^{2\sigma} < \frac{1}{2},$$

then $\mathcal{T}$ maps $B_R \to B_R$. A similar estimate shows:

$$\|\mathcal{T}[\psi] - \mathcal{T}[\phi]\|_{X_T} \le CT^{\gamma}R^{2\sigma}\|\psi - \phi\|_{X_T},$$

which makes $\mathcal{T}$ a contraction. By the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point in X_T. □

3.2. Global existence in the defocusing (q,τ)-fractional Schrödinger equation

We consider the defocusing nonlinear (q,τ)-fractional Schrödinger equation:

$$i^{\gamma}\,{}^{C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = -\Delta\psi + V(x)\psi + \lambda|\psi|^{2\sigma}\psi, \quad \lambda > 0,\ 0 < \gamma < 1,$$

with initial data $\psi_0 \in H^1(\Omega)$, and $\sigma < \frac{2}{d-2}$ (energy-subcritical).

Proposition 6 (Global Existence: Defocusing Case)**.** *Let $\psi_0 \in H^1(\Omega)$, $\lambda > 0$, $0 < \gamma < 1$, and $\sigma < \frac{2}{d-2}$. Then the nonlinear (q,τ)-fractional Schrödinger equation admits a unique global mild solution*

$$\psi \in C([0,\infty); H^1(\Omega)).$$

Proof. We proceed in three steps:
Step 1: Local Existence. From the previous proposition, we already know that a unique local mild solution exists in the time interval $[0,T]$ for some $T > 0$.
Step 2: A Priori Bounds. We define the energy-like functional:

$$\mathcal{E}(t) := \frac{1}{2}\|\nabla\psi(t)\|^2_{L^2} + \frac{1}{2}\int_\Omega V(x)|\psi(x,t)|^2dx + \frac{\lambda}{\sigma+1}\|\psi(t)\|^{2\sigma+2}_{L^{2\sigma+2}}.$$

For $\lambda > 0$ and $V(x) \geq 0$, all terms in $\mathcal{E}(t)$ are non-negative. Since the (q,τ)-Caputo fractional derivative ${}^{C}\nabla^{\gamma}_{q,\tau}\psi$ introduces memory effects but no gain of energy, the dissipation of norm over time implies:

$$\|\psi(t)\|_{L^2} \leq \|\psi_0\|_{L^2}, \quad \forall t \geq 0.$$

Furthermore, we apply a Gagliardo-Nirenberg inequality:

$$\|\psi\|^{2\sigma+2}_{L^{2\sigma+2}} \leq C\|\nabla\psi\|^{d\sigma}_{L^2}\|\psi\|^{2+2\sigma-d\sigma}_{L^2}.$$

Because $\|\psi(t)\|_{L^2}$ is bounded and $\sigma < \frac{2}{d-2}$, the energy remains controlled:

$$\mathcal{E}(t) \leq C\left(\|\nabla\psi(t)\|^2_{L^2} + \|\nabla\psi(t)\|^{d\sigma}_{L^2}\right),$$

which leads to an a priori estimate on $\|\nabla\psi(t)\|_{L^2}$.
Step 3: Continuation. By contradiction, suppose the maximal solution blows up at some $T_{\max} < \infty$. Then the H^1-norm must become unbounded:

$$\limsup_{t\to T^-_{\max}} \|\psi(t)\|_{H^1} = \infty.$$

However, the energy estimate and conservation of L^2 norm yield a contradiction. Hence, the solution can be extended globally. □

3.3. Role of (q,τ)-Hermite polynomials in fractional quantum models

Let us consider the generalized (q,τ)-Hermite polynomials defined as:

$$H^{(\tau)}_n(x|q) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{[n]_q!}{[k]_q![n-2k]_q!}\cdot\frac{\Gamma_{q,\tau}(\alpha+n-2k)}{\Gamma_{q,\tau}(\alpha)}x^{n-2k},$$

where $[n]_q = \frac{1-q^n}{1-q}$, and $\Gamma_{q,\tau}(\cdot)$ is the (q,τ)-Gamma function. We now expand the wave function $\psi(x,t)$ as:

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n(t)H^{(\tau)}_n(x|q),$$

and substitute this into the nonlinear (q,τ)-fractional Schrödinger equation:

$$i^{\gamma C}\nabla^{\gamma}_{q,\tau}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t) + \lambda|\psi(x,t)|^{2\sigma}\psi(x,t).$$

Assuming $\hat{H}_{q,\tau}H_n^{(\tau)} = E_n^{(q,\tau)}H_n^{(\tau)}$, with spectral values

$$E_n^{(q,\tau)} = \frac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)},$$

we obtain a system of nonlinear fractional ODEs for the modal coefficients:

$$i^{\gamma\,C}\nabla_{q,\tau}^{\gamma}a_n(t) = E_n^{(q,\tau)}a_n(t) + \lambda\sum_{k,j,l}C_{nkjl}\bar{a}_k(t)a_j(t)a_l(t),$$

where $C_{nkjl} = \int_{-1}^{1}H_n^{(\tau)}H_k^{(\tau)}H_j^{(\tau)}H_l^{(\tau)}w^{(\tau)}(x;q)dx$. The use of (q,τ)-Hermite polynomials embeds the quantum deformation and memory directly into the modal structure. The spectrum $E_n^{(q,\tau)}$ reflects both nonlocality and quantum geometry, and modifies the dispersion relation. The nonlinearity introduces modal coupling, but the fractional damping stabilizes high-energy modes under refocusing interaction. In the linear case ($\lambda = 0$), we obtain:

$$a_n(t) = a_n(0)\cdot E_{\gamma}^{(q,\tau)}\left(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}\right),$$

so each spectral component evolves independently via a deformed Mittag-Leffler function. These polynomials can be used in a Galerkin spectral method for numerically solving the full system:

$$\psi_N(x,t) = \sum_{n=0}^{N}a_n(t)H_n^{(\tau)}(x|q),$$

leading to high-accuracy simulations that capture both memory and deformation effects.

3.3.1. Existence of solutions using (q,τ)-Hermite polynomial expansion

We consider the nonlinear (q,τ)-fractional Schrödinger equation:

$$i^{\gamma\,C}\nabla_{q,\tau}^{\gamma}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t) + \lambda|\psi(x,t)|^{2\sigma}\psi(x,t), \quad \psi(x,0) = \psi_0(x), \tag{3}$$

with fractional order $0 < \gamma < 1$, $\lambda \in \mathbb{R}$, and $\sigma > 0$. We assume that the (q,τ)-Hermite polynomials $\{H_n^{(\tau)}(x|q)\}$ form a complete orthogonal basis in $L^2([-1,1], w^{(\tau)}(x;q)dx)$ with corresponding eigenvalues:

$$\hat{H}_{q,\tau}H_n^{(\tau)}(x|q) = E_n^{(q,\tau)}H_n^{(\tau)}(x|q), \quad E_n^{(q,\tau)} = \frac{\Gamma_{q,\tau}(\alpha+n)}{\Gamma_{q,\tau}(\alpha)}.$$

Proposition 7 (Existence via (q,τ)-Hermite Modal Expansion)**.** *Let $\psi_0(x) \in L^2([-1,1], w^{(\tau)}(x;q)dx)$ and $\lambda \in \mathbb{R}$, $\sigma < \frac{2}{d}$. Then, there exists $T > 0$ and a unique solution $\psi(x,t)$ of* (3) *of the form:*

$$\psi(x,t) = \sum_{n=0}^{\infty}a_n(t)H_n^{(\tau)}(x|q),$$

where the coefficients $a_n(t)$ solve a system of fractional ODEs:

$$i^{\gamma\,C}\nabla_{q,\tau}^{\gamma}a_n(t) = E_n^{(q,\tau)}a_n(t) + \lambda\sum_{k,j,l=0}^{\infty}C_{nkjl}\bar{a}_k(t)a_j(t)a_l(t),$$

and $C_{nkjl} = \int_{-1}^{1}H_n^{(\tau)}H_k^{(\tau)}H_j^{(\tau)}H_l^{(\tau)}w^{(\tau)}(x;q)dx$. The series converges in $C([0,T];L^2)$.

Proof. We expand $\psi(x,t) = \sum_{n=0}^{\infty}a_n(t)H_n^{(\tau)}(x|q)$ and substitute into Eq. (3). Using the orthogonality of $\{H_n^{(\tau)}\}$ under $w^{(\tau)}(x;q)$, we obtain the infinite-dimensional system:

$$i^{\gamma\,C}\nabla_{q,\tau}^{\gamma}a_n(t) = E_n^{(q,\tau)}a_n(t) + \lambda\sum_{k,j,l=0}^{\infty}C_{nkjl}\bar{a}_k(t)a_j(t)a_l(t).$$

We define the Banach space:

$$X_T := \left\{ \{a_n(t)\}_{n=0}^{\infty} : \sup_{t\in[0,T]} \sum_{n=0}^{\infty} |a_n(t)|^2 < \infty \right\}.$$

We apply the contraction mapping principle on the operator:

$$(\mathcal{T}\vec{a})_n(t) := a_n(0)E_{\gamma}^{(q,\tau)}(-i^{\gamma}E_n^{(q,\tau)}t^{\gamma}) + \lambda \int_0^t (t-s)^{\gamma-1} \sum_{k,j,l} C_{nkjl}\bar{a}_k(s)a_j(s)a_l(s)\, ds.$$

The bilinear form is locally Lipschitz due to boundedness of C_{nkjl} and the subcritical condition $\sigma < \frac{2}{d}$. Fractional integrals preserve continuity on $[0,T]$. Thus, $\mathcal{T}$ is a contraction on a closed ball in X_T for small T. By Banach's fixed-point theorem, there exists a unique fixed point $\vec{a}(t)$ in X_T, giving a unique mild solution $\psi(x,t)$ in $C([0,T];L^2)$. □

3.3.2. Energy functional in modal form

The energy-like functional associated with the modal expansion is:

$$\mathbb{E}(t) := \sum_{n=0}^{\infty} E_n^{(q,\tau)}|a_n(t)|^2 + \frac{\lambda}{\sigma+1} \sum_{n,j,k,l=0}^{\infty} D_{njkl}\bar{a}_n(t)a_j(t)a_k(t)a_l(t),$$

where $D_{njkl} := \int_{-1}^{1} H_n^{(\tau)}H_j^{(\tau)}H_k^{(\tau)}H_l^{(\tau)}w^{(\tau)}(x;q)dx$ represents the modal contribution of the nonlinearity. This energy is conserved for $\gamma = 1$, and decays algebraically in time when $0 < \gamma < 1$ due to the memory effect introduced by the (q,τ)-fractional derivative.

3.3.3. Lyapunov-type decay estimate

We define a Lyapunov-type functional:

$$\mathcal{L}(t) := \sum_{n=0}^{\infty} E_n^{(q,\tau)}|a_n(t)|^2.$$

Differentiating in the (q,τ)-Caputo sense and using the evolution equation for $a_n(t)$, we obtain:

$${}^{C}\nabla_{q,\tau}^{\gamma}\mathcal{L}(t) \le -c\sum_{n=0}^{\infty}[E_n^{(q,\tau)}]^2|a_n(t)|^2 + C\lambda\|a(t)\|^4,$$

where $c > 0$ depends on the spectral gap and C depends on bounds for the nonlinear terms. For defocusing nonlinearity ($\lambda > 0$) and small initial data, this yields: $\mathcal{L}(t) \le C_0(1+t)^{-\gamma}$, indicating algebraic decay of energy and stability of the zero solution.

Lemma 1 (Conditional generalized (q,τ)-Grönwall inequality and algebraic decay)**.** *Let*

$$0 < q < 1, \qquad \tau > 0, \qquad 0 < \gamma < 1, \qquad T > 0.$$

Let $\mathcal{D}_{q,\tau}^{\gamma} := {}^{C}\nabla_{q,\tau}^{\gamma}$ denote the Caputo-type (q,τ)-fractional operator considered in this paper. Assume that the operator is defined on a real vector space $\mathcal{X}_T \subset C([0,T])$ and satisfies the following hypotheses:

(H1) ***Linearity and admissibility.*** *The operator is linear on $\mathcal{X}_T$, and $\mathcal{D}_{q,\tau}^{\gamma}c = 0$ for every constant function c. All functions appearing below, including the scalar relaxation solutions, belong to $\mathcal{X}_T$.*

(H2) ***First-contact principle.*** *For every $v \in \mathcal{X}_T$ and $t_* \in (0,T]$, if*

$$v(t) < v(t_*) \quad \text{for all } 0 \le t < t_*,$$

then

$$\mathcal{D}_{q,\tau}^{\gamma}v(t_*) \ge 0.$$

(H3) ***Comparison principle.*** *For every $\mu > 0$, if $w \in \mathcal{X}_T$ satisfies*

$$\mathcal{D}_{q,\tau}^{\gamma}w(t) + \mu w(t) \le 0, \qquad 0 < t \le T,$$

and

$$w(0) \le 0,$$

then

$$w(t) \leq 0, \qquad 0 \leq t \leq T.$$

(H4) ***Positive scalar relaxation.*** *For every $\mu > 0$, the initial-value problem*

$$\begin{cases} \mathcal{D}^{\gamma}_{q,\tau} S_\mu(t) + \mu S_\mu(t) = 0, \quad t > 0, \\ S_\mu(0) = 1, \end{cases}$$

admits a solution satisfying

$$S_\mu(t) \geq 0, \qquad t \geq 0,$$

and

$$S_\mu(t) \leq \frac{K_{q,\tau,\gamma,\mu}}{1+t^{\gamma}}, \qquad t \geq 0,$$

where $K_{q,\tau,\gamma,\mu} > 0$ is independent of the final time T.

Let $u \in \mathcal{X}_T$ be nonnegative and satisfy

$$\mathcal{D}^{\gamma}_{q,\tau} u(t) \leq -au(t) + bu(t)^p, \qquad 0 < t \leq T, \tag{4}$$

where

$$a > 0, \qquad b \geq 0, \qquad 1 < p < 2.$$

Suppose that there exists $\eta > 0$ such that

$$0 \leq u(0) < \eta, \qquad \mu := a - b\eta^{p-1} > 0. \tag{5}$$

Then

$$0 \leq u(t) \leq u(0) S_\mu(t) \leq \frac{K_{q,\tau,\gamma,\mu} u(0)}{1+t^{\gamma}}, \qquad 0 \leq t \leq T. \tag{6}$$

In particular,

$$u(t) \leq K_{q,\tau,\gamma,\mu} u(0)(1+t)^{-\gamma}. \tag{7}$$

If u is globally defined and hypotheses (H1)–(H4) *hold on every finite interval with constants independent of T, then* $\lim_{t\to\infty} u(t) = 0$.

Proof. We divide the proof into three steps.

Step 1. Establishing an invariant smallness bound. We first prove that

$$u(t) < \eta, \qquad 0 \leq t \leq T. \tag{8}$$

Suppose, to the contrary, that there exists a time $t \in (0, T]$ for which $u(t) \geq \eta$. Since u is continuous and $u(0) < \eta$, there exists a first contact time $t_* \in (0, T]$ satisfying $u(t_*) = \eta$, and $u(t) < \eta$, $0 \leq t < t_*$. In view of (H2), we have

$$\mathcal{D}^{\gamma}_{q,\tau} u(t_*) \geq 0. \tag{9}$$

However, using (4), we receive that

$$\begin{aligned} \mathcal{D}^{\gamma}_{q,\tau} u(t_*) &\leq -au(t_*) + bu(t_*)^p \\ &= -a\eta + b\eta^p \\ &= -\eta(a - b\eta^{p-1}) \\ &= -\mu\eta < 0. \end{aligned} \tag{10}$$

This contradicts (9). Therefore, this gives

$$0 \leq u(t) < \eta, \qquad 0 \leq t \leq T.$$

Step 2. Reduction to a linear fractional inequality. Since $u(t) \geq 0$ and $p > 1$, the invariant estimate implies

$$u(t)^p = u(t)^{p-1} u(t) \leq \eta^{p-1} u(t).$$

Consequently,

$$\begin{aligned}\mathcal{D}^{\gamma}_{q,\tau}u(t) &\leq -au(t) + bu(t)^p \\ &\leq -au(t) + b\eta^{p-1}u(t) \\ &= -(a - b\eta^{p-1})u(t) \\ &= -\mu u(t).\end{aligned}$$

Hence, this leads to

$$\mathcal{D}^{\gamma}_{q,\tau}u(t) + \mu u(t) \leq 0, \qquad 0 < t \leq T. \tag{11}$$

Step 3. Application of the comparison principle. Let S_μ be the nonnegative relaxation solution in (H4). Define

$$w(t) = u(t) - u(0)S_\mu(t).$$

By (H1), w belongs to the admissible operator domain. Moreover, $w(0) = u(0) - u(0)S_\mu(0) = 0$. Using the linearity of the fractional operator, we obtain

$$\mathcal{D}^{\gamma}_{q,\tau}w(t) + \mu w(t) = \mathcal{D}^{\gamma}_{q,\tau}u(t) + \mu u(t) - u(0)\left[\mathcal{D}^{\gamma}_{q,\tau}S_\mu(t) + \mu S_\mu(t)\right].$$

Since $\mathcal{D}^{\gamma}_{q,\tau}S_\mu(t) + \mu S_\mu(t) = 0$, and by (11), we conclude that

$$\mathcal{D}^{\gamma}_{q,\tau}w(t) + \mu w(t) \leq 0.$$

Applying the comparison principle (H3) gives

$$w(t) \leq 0, \qquad 0 \leq t \leq T.$$

Therefore,

$$0 \leq u(t) \leq u(0)S_\mu(t).$$

Now hypothesis (H4) yields

$$0 \leq u(t) \leq \frac{K_{q,\tau,\gamma,\mu}u(0)}{1 + t^\gamma}. \tag{12}$$

Furthermore, for $0 < \gamma < 1$, the subadditivity inequality gives

$$(1 + t)^\gamma \leq 1 + t^\gamma, \qquad t \geq 0.$$

Thus, we get

$$\frac{1}{1 + t^\gamma} \leq (1 + t)^{-\gamma}.$$

It follows that $u(t) \leq K_{q,\tau,\gamma,\mu}u(0)(1 + t)^{-\gamma}$. Finally, if u exists globally and the stated hypotheses hold uniformly on every finite interval, then $\lim_{t\to\infty} u(t) = 0$. □

Assumptions for the infinite-dimensional modal problem. Let $\{\phi_n\}_{n\geq 0}$ be an independently established orthonormal eigenbasis of a Hilbert space $\mathcal{H}$, with corresponding eigenvalues $\lambda_n \geq 0$. For $s \geq 0$, define

$$X_s = \left\{ a = (a_n)_{n\geq 0} : \|a\|^2_{X_s} := \sum_{n=0}^{\infty} (1 + \lambda_n)^s |a_n|^2 < \infty \right\}.$$

Write $w_n = (1 + \lambda_n)^s$ and define

$$\mathcal{N}_n(a) = \sum_{k,j,\ell=0}^{\infty} C_{nkj\ell}\, \overline{a_k} a_j a_\ell.$$

Assume that the interaction coefficients satisfy

$$M_s^2 := \sum_{n,k,j,\ell=0}^{\infty} \frac{w_n |C_{nkj\ell}|^2}{w_k w_j w_\ell} < \infty. \tag{13}$$

This is a sufficient summability condition; it is not asserted to be necessary.

Passage to the infinite-dimensional limit. Let P_N denote projection onto modes $0, \dots, N$. The initial approximations are $P_N a^0$, where $a^0 \in X_s$, so that

$$P_N a^0 \longrightarrow a^0 \quad \text{in } X_s.$$

Assume that the linear evolution admits the mild form

$$a(t) = S(t)a^0 + \int_0^t R(t-r)\mathcal{N}(a(r))\, dr,$$

where $S(t)$ is strongly continuous, $S(0) = I$, and the modal projections commute with $S(t)$ and $R(t)$. For every finite $T > 0$, assume

$$\sup_{0 \le t \le T} \|S(t)\|_{\mathcal{L}(X_s)} < \infty, \qquad \|R(t)\|_{\mathcal{L}(X_s)} \le K_T t^{\gamma-1}, \quad 0 < t \le T,$$

with $0 < \gamma < 1$ and constants independent of N. These bounds and Lemma **??** give a common local existence interval by contraction. Strong convergence of P_N and the same contraction estimates yield convergence of the Galerkin solutions in $C([0, T_*]; X_s)$ on that interval. Continuation to arbitrary finite times requires an a priori bound for $\|a(t)\|_{X_s}$ on every finite interval, uniform in the truncation.

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with $0 < \gamma < 1$ and constants independent of N. These bounds and Lemma **??** give a common local existence interval by contraction. Strong convergence of P_N and the same contraction estimates yield convergence of the Galerkin solutions in $C([0, T_*]; X_s)$ on that interval. Continuation to arbitrary finite times requires an a priori bound for $\|a(t)\|_{X_s}$ on every finite interval, uniform in the truncation.

Proposition 8 (Asymptotic Stability)**.** *Under the assumptions of the Lyapunov decay estimate, the zero solution $\psi(x,t) \equiv 0$ of the (q,τ)-fractional Schrödinger equation is asymptotically stable. That is, for sufficiently small initial data ψ_0, $\lim_{t\to\infty} \|\psi(t)\|_{L^2} = 0$, and the modal energy satisfies $\lim_{t\to\infty} \sum_{n=0}^{\infty} E_n^{(q,\tau)} |a_n(t)|^2 = 0$.*

Proof. Let $\psi_0 \in L^2([-1,1], w^{(\tau)}(x;q)dx)$ be the initial data with modal expansion $\psi_0(x) = \sum_{n=0}^{\infty} a_n(0) H_n^{(\tau)}(x|q)$, and suppose $\psi(x,t) = \sum_{n=0}^{\infty} a_n(t) H_n^{(\tau)}(x|q)$ is the solution. Define the modal Lyapunov functional:

$$\mathcal{L}(t) = \sum_{n=0}^{\infty} E_n^{(q,\tau)} |a_n(t)|^2.$$

Applying the (q,τ)-Caputo derivative:

$${}^{C}\nabla_{q,\tau}^{\gamma} \mathcal{L}(t) = 2\Re \sum_{n=0}^{\infty} E_n^{(q,\tau)} \bar{a}_n(t) {}^{C}\nabla_{q,\tau}^{\gamma} a_n(t).$$

Using the evolution equation:

$${}^{C}\nabla_{q,\tau}^{\gamma} a_n(t) = -i^{\gamma} E_n^{(q,\tau)} a_n(t) - i^{\gamma} \lambda \sum_{k,j,l} C_{nkjl} \bar{a}_k(t) a_j(t) a_l(t),$$

we obtain

$$ {}^{C}\nabla^{\gamma}_{q,\tau}\mathcal{L}(t) = -2\Im(i^{\gamma})\sum_{n=0}^{\infty}[E_n^{(q,\tau)}]^2|a_n(t)|^2 + \mathcal{N}(t), $$

where $\mathcal{N}(t)$ is a nonlinear term given by

$$ \begin{aligned} \mathcal{N}(t) &:= -2\lambda\Im(i^{\gamma})\sum_{n=0}^{\infty}E_n^{(q,\tau)}\sum_{k,j,l=0}^{\infty}C_{nkjl}\bar{a}_n(t)\bar{a}_k(t)a_j(t)a_l(t) \\ &= -2\lambda\Im(i^{\gamma})\sum_{n,k,j,l=0}^{\infty}E_n^{(q,\tau)}C_{nkjl}\bar{a}_n(t)\bar{a}_k(t)a_j(t)a_l(t). \end{aligned} $$

For $\gamma \in (0,1)$, $\Im(i^{\gamma}) > 0$. Assume $\lambda > 0$ and $\|a(t)\|$ small, so the nonlinear term is dominated. Then, we get

$$ {}^{C}\nabla^{\gamma}_{q,\tau}\mathcal{L}(t) \leq -c\mathcal{L}(t) + \varepsilon\mathcal{L}(t)^2. $$

This yields an inequality of the form:

$$ {}^{C}\nabla^{\gamma}_{q,\tau}\mathcal{L}(t) + c\mathcal{L}(t) \leq \varepsilon\mathcal{L}(t)^2. $$

Using standard fractional differential inequality techniques (see Lemma **??**), one can deduce:

$$ \mathcal{L}(t) \leq C_0(1+t)^{-\gamma}, $$

proving that $\mathcal{L}(t) \to 0$ as $t \to \infty$. Hence, $\psi(x,t) \to 0$ in L^2-norm, which proves asymptotic stability. □

Remark 1 (Nonlinear Terms in the Energy Derivative)**.** *In the computation of*

$$ {}^{C}\nabla^{\gamma}_{q,\tau}\mathcal{L}(t) = -2\Im(i^{\gamma})\sum_{n=0}^{\infty}[E_n^{(q,\tau)}]^2|a_n(t)|^2 + \mathcal{N}(t), $$

the nonlinear terms arise from the cubic interaction in the evolution of $a_n(t)$:

$$ {}^{C}\nabla^{\gamma}_{q,\tau}a_n(t) = -i^{\gamma}E_n^{(q,\tau)}a_n(t) - i^{\gamma}\lambda\sum_{k,j,l=0}^{\infty}C_{nkjl}\bar{a}_k(t)a_j(t)a_l(t). $$

Substituting this into the derivative of $\mathcal{L}(t)$ yields the nonlinear contribution:

$$ \begin{aligned} \mathcal{N}(t) &= -2\lambda\Im(i^{\gamma})\sum_{n=0}^{\infty}E_n^{(q,\tau)}\sum_{k,j,l=0}^{\infty}C_{nkjl}\bar{a}_n(t)\bar{a}_k(t)a_j(t)a_l(t) \\ &= -2\lambda\Im(i^{\gamma})\sum_{n,k,j,l=0}^{\infty}E_n^{(q,\tau)}C_{nkjl}\bar{a}_n(t)\bar{a}_k(t)a_j(t)a_l(t). \end{aligned} $$

This term is a quartic form in the coefficients $a_n(t)$, modulated by the nonlinear coupling constants C_{nkjl} and the energy weights $E_n^{(q,\tau)}$.

Proposition 9 (Bound on Nonlinear Terms $\mathcal{N}(t)$)**.** *Let $\mathcal{N}_{nkjl}(t) := \bar{a}_n(t)\bar{a}_k(t)a_j(t)a_l(t)$ and assume there exists a constant $C_0 > 0$ such that $|C_{nkjl}| \leq C_0$ uniformly in indices. Then the nonlinear part of the energy derivative satisfies:*

$$ \left|\sum_{n,k,j,l=0}^{\infty}E_n^{(q,\tau)}C_{nkjl}\mathcal{N}_{nkjl}(t)\right| \leq C_0\left(\sum_{n=0}^{\infty}E_n^{(q,\tau)}|a_n(t)|^2\right)^2 = C_0\mathcal{L}(t)^2. $$

Thus, the nonlinear terms are quadratically bounded by the modal energy.

Proof. We estimate the quartic term in the nonlinear contribution to the energy derivative:

$$ \sum_{n,k,j,l=0}^{\infty}E_n^{(q,\tau)}C_{nkjl}\bar{a}_n(t)\bar{a}_k(t)a_j(t)a_l(t). $$

Taking absolute value and applying the triangle inequality:

$$\left| \sum_{n,k,j,l} E_n^{(q,\tau)} C_{nkjl} \bar{a}_n \bar{a}_k a_j a_l \right| \leq \sum_{n,k,j,l} |E_n^{(q,\tau)}||C_{nkjl}||a_n||a_k||a_j||a_l|.$$

Assume $|C_{nkjl}| \leq C_0$ for all indices. Then:

$$\leq C_0 \sum_{n,k,j,l} E_n^{(q,\tau)} |a_n||a_k||a_j||a_l|.$$

Group the sum as:

$$= C_0 \left(\sum_n E_n^{(q,\tau)} |a_n| \right) \left(\sum_k |a_k| \right) \left(\sum_j |a_j| \right) \left(\sum_l |a_l| \right).$$

Applying Cauchy-Schwarz to each factor:

$$\sum_n E_n^{(q,\tau)} |a_n| \leq \left(\sum_n E_n^{(q,\tau)} |a_n|^2 \right)^{1/2} \left(\sum_n E_n^{(q,\tau)} \right)^{1/2},$$

$$\sum_k |a_k| \leq \left(\sum_k |a_k|^2 \right)^{1/2} \cdot \sqrt{N},$$

for finite-mode approximations. For convergence, we simplify to obtain:

$$\leq C_0' \left(\sum_n E_n^{(q,\tau)} |a_n|^2 \right)^2 = C_0' \mathcal{L}(t)^2,$$

where $C_0' = C_0(\sum_n E_n^{(q,\tau)})$ is finite under truncation or decay assumptions. □

Example 3 (Truncated Expansion: $N = 2$ Modes)**.** *Suppose the solution is approximated using the first 3 modes:*

$$\psi(x,t) \approx a_0(t) H_0^{(\tau)}(x|q) + a_1(t) H_1^{(\tau)}(x|q) + a_2(t) H_2^{(\tau)}(x|q).$$

The nonlinear term becomes:

$$\sum_{n,k,j,l=0}^{2} E_n^{(q,\tau)} C_{nkjl} \bar{a}_n(t) \bar{a}_k(t) a_j(t) a_l(t),$$

which is a finite sum of 81 quarter terms. If $C_{nkjl} \leq 1$ and $E_n^{(q,\tau)} \leq E_2$, then:

$$|\mathcal{N}(t)| \leq E_2 \left(\sum_{n=0}^{2} |a_n(t)|^2 \right)^2.$$

Hence, the nonlinear part is bounded by $E_2 \|\vec{a}(t)\|_2^4$, where $\vec{a}(t) = (a_0(t), a_1(t), a_2(t))$.

Uniform Galerkin estimates and passage to the infinite-dimensional limit

Let $\{\phi_n\}_{n\geq 0}$ be an orthonormal eigenbasis of the Hilbert space $\mathcal{H}$, with eigenvalues $\lambda_n \geq 0$. For $s \geq 0$, define

$$X_s = \left\{ a = (a_n)_{n\geq 0} : \|a\|_{X_s}^2 = \sum_{n=0}^{\infty} (1+\lambda_n)^s |a_n|^2 < \infty \right\}.$$

Put $w_n = (1+\lambda_n)^s$. Consider the cubic modal nonlinearity

$$\mathcal{F}_n(a) = \sum_{k,j,\ell=0}^{\infty} C_{nkj\ell} \overline{a_k} a_j a_\ell.$$

Assume the weighted summability condition

$$M_s^2 := \sum_{n,k,j,\ell=0}^{\infty} \frac{w_n |C_{nkj\ell}|^2}{w_k w_j w_\ell} < \infty. \tag{14}$$

Let P_N be the orthogonal projection onto modes $0, \dots, N$, and define $\mathcal{F}_N(a) = P_N \mathcal{F}(P_N a)$.

Proposition 10 (Uniform nonlinear Galerkin estimates)**.** *Under condition* (14)*, the nonlinear mapping $\mathcal{F} : X_s \to X_s$ is well-defined and satisfies*

$$\|\mathcal{F}(a)\|_{X_s} \le M_s \|a\|_{X_s}^3. \tag{15}$$

Furthermore, for all $a, b \in X_s$,

$$\|\mathcal{F}(a) - \mathcal{F}(b)\|_{X_s} \le M_s \left(\|a\|_{X_s}^2 + \|a\|_{X_s} \|b\|_{X_s} + \|b\|_{X_s}^2 \right) \|a - b\|_{X_s}. \tag{16}$$

The same estimates hold for $\mathcal{F}_N$, with constants independent of N.

Proof. For every fixed n, the Cauchy–Schwarz inequality gives

$$|\mathcal{F}_n(a)|^2 \le \left(\sum_{k,j,\ell=0}^{\infty} \frac{|C_{nkj\ell}|^2}{w_k w_j w_\ell} \right) \times \left(\sum_{k,j,\ell=0}^{\infty} w_k w_j w_\ell |a_k|^2 |a_j|^2 |a_\ell|^2 \right).$$

The second factor equals

$$\left(\sum_{k=0}^{\infty} w_k |a_k|^2 \right)^3 = \|a\|_{X_s}^6.$$

Multiplying by w_n and summing over n, we obtain

$$\begin{aligned} \|\mathcal{F}(a)\|_{X_s}^2 &= \sum_{n=0}^{\infty} w_n |\mathcal{F}_n(a)|^2 \\ &\le M_s^2 \|a\|_{X_s}^6. \end{aligned}$$

Taking square roots proves (15). For the difference estimate, write

$$\overline{a_k} a_j a_\ell - \overline{b_k} b_j b_\ell = \overline{(a_k - b_k)} a_j a_\ell + \overline{b_k} (a_j - b_j) a_\ell + \overline{b_k} b_j (a_\ell - b_\ell).$$

Applying the preceding weighted Cauchy–Schwarz estimate separately to these three terms yields

$$\|\mathcal{F}(a) - \mathcal{F}(b)\|_{X_s} \le M_s \big(\|a\|_{X_s}^2 + \|a\|_{X_s} \|b\|_{X_s} + \|b\|_{X_s}^2 \big) \|a - b\|_{X_s}.$$

Finally, since P_N is an orthogonal coordinate projection in X_s, $\|P_N a\|_{X_s} \le \|a\|_{X_s}$. It follows that $\|\mathcal{F}_N(a)\|_{X_s} \le M_s \|a\|_{X_s}^3$, and the same Lipschitz estimate holds for $\mathcal{F}_N$. The constant M_s is independent of N, proving the assertion. □

Theorem 3 (Conditional global existence and Galerkin convergence for small data)**.** *Let $0 < q < 1$, $\tau > 0$, and $0 < \gamma < 1$. Let $s \ge 0$, $a^0 \in X_s$, and assume that the summability condition* (14) *holds. Suppose that the modal evolution equation has the mild formulation*

$$a(t) = S(t) a^0 + \int_0^t R(t - r) \mathcal{F}(a(r))\, dr, \tag{17}$$

where $S(t)$ is strongly continuous, $S(0) = I$, and $S(t)$ and $R(t)$ commute with the modal projections. Assume that for every $T > 0$, $\sup_{0 \le t \le T} \|S(t)\|_{\mathcal{L}(X_s)} \le A_T$, and

$$\|R(t)\|_{\mathcal{L}(X_s)} \le B_T t^{\gamma - 1}, \qquad 0 < t \le T,$$

where A_T, B_T do not depend on the truncation level N. Let $a^{(N)}$ solve the truncated mild equation

$$a^{(N)}(t) = S(t) P_N a^0 + \int_0^t R(t - r) P_N \mathcal{F}(a^{(N)}(r))\, dr,$$

with $a^{(N)} = P_N a^{(N)}$. Finally, assume there exists $\varepsilon > 0$ such that whenever $\|a^0\|_{X_s} \le \varepsilon$, the truncated solutions satisfy, for every finite $T > 0$,

$$\sup_{N\ge 0}\sup_{0\le t\le T}\|a^{(N)}(t)\|_{X_s} \le C_T, \tag{18}$$

where C_T is independent of N, and the same bound holds on every interval on which the truncated solution exists. Then the truncated solutions exist globally and converge in $C([0,T];X_s)$, for every finite $T > 0$, to a unique global mild solution $a \in C([0,\infty);X_s)$. Moreover, the limiting solution satisfies $\sup_{0\le t\le T}\|a(t)\|_{X_s} \le C_T$. If, in addition, the modal energy $L(t) = \sum_{n=0}^{\infty} E_n^{(q,\tau)}|a_n(t)|^2$ satisfies the assumptions of Lemma 1, and the energy weights obey

$$E_n^{(q,\tau)} \ge c_s(1+\lambda_n)^s, \qquad c_s > 0,$$

then the corresponding conditional energy-decay estimate also controls the X_s-norm of the solution.

Proof. By Proposition 10, the mappings $\mathcal{F}_N$ are locally Lipschitz on X_s with constants independent of N. On a ball of radius R,

$$\|\mathcal{F}_N(a) - \mathcal{F}_N(b)\|_{X_s} \le 3M_s R^2\|a-b\|_{X_s}.$$

Using the mild formulation, the Lipschitz constant of the associated fixed-point map on $C([0,h];X_s)$ is bounded by $\frac{3B_h M_s R^2 h^\gamma}{\gamma}$. For sufficiently small $h > 0$, this quantity is less than one. The contraction mapping theorem therefore gives local solutions on a common interval, independent of N. The bound (18) prevents the truncated solutions from leaving a bounded subset of X_s on any finite interval. Since the truncated nonlinear systems are locally Lipschitz, the continuation principle extends each solution to every finite time. We next establish convergence. Let a denote the unique local solution of the infinite-dimensional mild equation, obtained by the same contraction argument. Because $P_N a^0 \to a^0$ in X_s, the initial-data error tends to zero. Subtracting the mild equations for $a^{(N)}$ and a gives

$$\begin{aligned}\|a^{(N)}(t) - a(t)\|_{X_s} \le{}& A_T\|(I-P_N)a^0\|_{X_s}\\ &+ B_T\int_0^t (t-r)^{\gamma-1}\|\mathcal{F}(a^{(N)}(r)) - \mathcal{F}(a(r))\|_{X_s}\,dr\\ &+ B_T\int_0^t (t-r)^{\gamma-1}\|(I-P_N)\mathcal{F}(a(r))\|_{X_s}\,dr.\end{aligned}$$

On any common bounded interval, the nonlinear Lipschitz estimate controls the second term by

$$3B_T M_s R_T^2\int_0^t (t-r)^{\gamma-1}\|a^{(N)}(r) - a(r)\|_{X_s}\,dr,$$

where R_T bounds the two solutions. Since $\mathcal{F}(a(r))$ is continuous in X_s, the set $\{\mathcal{F}(a(r)) : 0 \le r \le T\}$ is compact in X_s. The projections P_N converge strongly to the identity and are uniformly bounded. Therefore,

$$\sup_{0\le r\le T}\|(I-P_N)\mathcal{F}(a(r))\|_{X_s} \longrightarrow 0.$$

The fractional integral comparison estimate now implies

$$\sup_{0\le t\le T}\|a^{(N)}(t) - a(t)\|_{X_s} \longrightarrow 0$$

on the common interval of existence. The uniform a priori bound permits the local convergence argument to be repeated over successive finite time intervals. Consequently, the limiting solution exists on $[0,\infty)$ and

$$a^{(N)} \longrightarrow a \quad \text{in } C([0,T];X_s)$$

for every finite $T > 0$. Passing to the limit in the uniform bound gives $\sup_{0\le t\le T}\|a(t)\|_{X_s} \le C_T$. Uniqueness follows from the local Lipschitz estimate and the fractional integral comparison inequality. Finally, if the modal energy satisfies the additional assumptions in the theorem,

then $L(t) \geq c_s \|a(t)\|^2_{X_s}$. Hence, we have $\|a(t)\|^2_{X_s} \leq c_s^{-1} L(t)$, and the conditional energy-decay estimate of Lemma 1 provides the corresponding X_s-bound. □

Theorem 4 (Existence in Sobolev Space H^s)**.** *Let $\psi_0 \in H^s([-1,1], w^{(\tau)}(x;q)dx)$ for some $s > 0$, and suppose the assumptions of the global existence theorem hold. Then the solution $\psi(x,t)$ of the (q,τ)-fractional Schrödinger equation satisfies:*

$$\psi(\cdot,t) \in H^s([-1,1], w^{(\tau)}(x;q)dx), \quad \text{for all } t \geq 0.$$

Moreover, for any $T > 0$, the map $t \mapsto \psi(\cdot,t)$ lies in the space:

$$C([0,T]; H^s([-1,1], w^{(\tau)}(x;q)dx)) \cap C^1((0,T]; H^{s-2}).$$

Proof. The modal expansion

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n(t) H_n^{(\tau)}(x|q)$$

converges in H^s if

$$\sum_{n=0}^{\infty} |a_n(t)|^2 (1+\lambda_n)^s < \infty,$$

where λ_n is the eigenvalue growth parameter (e.g., $E_n^{(q,\tau)}$). From the global decay result, $|a_n(t)| \leq C_n(1+t)^{-\gamma/2}$, and the regularity of the $H_n^{(\tau)}$ basis, we deduce that $\psi(x,t)$ inherits the regularity of ψ_0 in H^s, and remains in H^s for all t. Differentiability in time follows from the regularity of the fractional derivative and the well-posedness of the modal ODEs in the H^s topology. □

Example 4 (Existence in Sobolev Space H^s for Modal Solutions)**.** *Consider the initial-value problem*

$$i^\gamma \, {}^C\nabla^\gamma_{q,\tau} \psi(x,t) = \hat{H}_{q,\tau} \psi(x,t), \quad \psi(x,0) = \psi_0(x),$$

where $\hat{H}_{q,\tau}$ is diagonalized by the orthonormal basis $\{H_n^{(\tau)}(x|q)\}_{n\geq 0}$ with eigenvalues $\{E_n^{(q,\tau)}\}$, and $\psi_0 \in H^s([-1,1])$, $s > 0$. We assume a modal expansion:

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n(t) H_n^{(\tau)}(x|q), \quad a_n(0) = a_n^{(0)} = \langle \psi_0, H_n^{(\tau)} \rangle.$$

Let $\mathcal{H}^s$ denote the modal Sobolev space:

$$\mathcal{H}^s := \left\{ \vec{a}(t) = \{a_n(t)\}_{n=0}^{\infty} : \sum_{n=0}^{\infty} (1 + E_n^{(q,\tau)})^s |a_n(t)|^2 < \infty \right\}.$$

Claim: *If $\psi_0 \in H^s$, then the solution $\psi(x,t) \in H^s$ for all $t \geq 0$, and the Sobolev norm remains bounded. The modal coefficients satisfy:*

$${}^C\nabla^\gamma_{q,\tau} a_n(t) = -i^\gamma E_n^{(q,\tau)} a_n(t), \quad a_n(0) = a_n^{(0)}.$$

Hence, we have

$$a_n(t) = a_n^{(0)} \, E_\gamma^{(q,\tau)}(-i^\gamma E_n^{(q,\tau)} t^\gamma),$$

where $E_\gamma^{(q,\tau)}(\cdot)$ is the (q,τ)-Mittag-Leffler function. Now estimate the Sobolev norm:

$$\|\psi(\cdot,t)\|^2_{H^s} \approx \sum_{n=0}^{\infty} (1 + E_n^{(q,\tau)})^s |a_n(t)|^2 = \sum_{n=0}^{\infty} (1 + E_n^{(q,\tau)})^s |a_n^{(0)}|^2 \left| E_\gamma^{(q,\tau)}(-i^\gamma E_n^{(q,\tau)} t^\gamma) \right|^2.$$

Using the uniform bound:

$$\left| E_\gamma^{(q,\tau)}(-i^\gamma E_n^{(q,\tau)} t^\gamma) \right| \leq \frac{C}{1 + E_n^{(q,\tau)} t^\gamma}, \quad \text{(from known decay properties)},$$

we get

$$\|\psi(\cdot,t)\|^2_{H^s} \leq C \sum_{n=0}^{\infty} (1 + E_n^{(q,\tau)})^{s-2} |a_n^{(0)}|^2$$

$$\leq C' \|\psi_0\|^2_{H^s}.$$

Thus, $\psi(x,t) \in H^s([-1,1])$ for all $t \geq 0$, and the solution is uniformly bounded in H^s.

Example 5 (Numerical Illustration for Different τ)**.** *We numerically investigate the decay of the Sobolev norm $\|\psi(\cdot,t)\|^2_{H^s}$ for the modal solution of the (q,τ)-fractional Schrödinger equation:*

$$i^\gamma \, {}^C\nabla^\gamma_{q,\tau}\psi(x,t) = \hat{H}_{q,\tau}\psi(x,t),$$

with deformation parameter $q = 0.5$, fractional order $\gamma = 0.8$, and Sobolev index $s = 1.5$. The wavefunction is approximated using the first $N = 5$ (q,τ)-Hermite modes with initial coefficients

$$a_n(0) = \frac{1}{(n+1)^{3/2}}, \quad n = 0, 1, \ldots, 4.$$

We simulate the solution for different values of the parameter $\tau \in \{0.8, 1.2, 2.0\}$, using the approximate modal expression:

$$a_n(t) \approx a_n(0) \cdot E^{(q,\tau)}_\gamma(-i^\gamma E^{(q,\tau)}_n t^\gamma),$$

where $E^{(q,\tau)}_n = \frac{1-q^{\tau(n+1)}}{1-q}$, and $E^{(q,\tau)}_\gamma$ is the deformed Mittag-Leffler function. This illustrates that tuning τ provides a way to modulate dissipation and long-term behavior in quantum systems modeled by fractional operators. The deformed system thus enables a flexible modeling tool for quantum processes with tunable memory and spectral decay rates.

Table 1. Numerical methodology for modal (q,τ)-Schrödinger simulation.

Parameter	***Value / Description***
Fractional Order γ	0.8
Deformation Parameter q	0.5
Memory Depth τ	$\{0.8, 1.2, 2.0\}$
Sobolev Index s	1.5
Modal Truncation N	5 *modes*
Time Domain	$[0, 10]$
Time Steps J	200
Mittag-Leffler Truncation K	20 *terms*
Initial Modal Coefficients	$a_n(0) = \frac{1}{(n+1)^{3/2}}$
Eigenvalues $E^{(q,\tau)}_n$	$\frac{1-q^{\tau(n+1)}}{1-q}$

The numerical results reveal that: ***Higher values of τ yield faster decay:*** *As τ increases, the spectral energy levels $E^{(q,\tau)}_n$ rise more rapidly, which accelerates the decay of the Mittag-Leffler kernel $E^{(q,\tau)}_\gamma(-i^\gamma E^{(q,\tau)}_n t^\gamma)$. This results in a more pronounced damping effect in the wave function evolution.* ***Memory effects are preserved for lower τ:*** *Smaller values of τ correspond to a slower decay rate, suggesting that the system retains more memory of its initial state. This is consistent with the interpretation of τ as a deformation parameter governing the temporal memory of the quantum system.* ***Uniform boundedness in H^s:*** *For all tested values of τ, the norm remains bounded, verifying the theoretical prediction that the solution $\psi(x,t)$ stays in the modal Sobolev space $\mathcal{H}^s$ over time.* ***Application insights:*** *These observations provide insight into the controllability and regularity of quantum systems modeled via (q,τ)-deformed operators. In practice, adjusting τ offers a tunable mechanism for modulating dissipation and temporal smoothness in quantum simulations. This simulation supports the utility of (q,τ)-Hermite expansions in fractional quantum dynamics and justifies their use in future numerical solvers and physical interpretations of nonlocal quantum models.*

Example 6 (Optical Beam Intensity for Different τ)**.** *We study the propagation of an optical beam governed by the (q,τ)-fractional Schrödinger equation in dimensionless form:*

$$i^\gamma \, {}^C\nabla^\gamma_{q,\tau}\psi(z) = \hat{H}_{q,\tau}\psi(z),$$

where $\psi(z)$ represents the beam profile along the propagation direction z, and the Hamiltonian $\hat{H}_{q,\tau}$ is diagonal in the (q,τ)-Hermite basis with eigenvalues

$$E_n^{(q,\tau)} = \frac{1-q^{\tau(n+1)}}{1-q}.$$

We consider the following parameters: $q = 0.6$, $\gamma = 0.9$, $n = 0,\ldots,4$, initial beam profile: $a_n(0) = \exp(-n^2/5)$, and propagation distance: $z \in [0.01, 10]$. The modal solution is approximated using the (q,τ)-Mittag-Leffler function:

$$a_n(z) \approx a_n(0) \cdot E_\gamma^{(q,\tau)}(-i^\gamma E_n^{(q,\tau)} z^\gamma),$$

and the optical intensity is computed as $|\psi(z)|^2 = \left|\sum_{n=0}^{4} a_n(z)\right|^2$. We simulate this beam evolution for three different values of the deformation parameter $\tau \in \{0.8, 1.1, 1.6\}$. Fig.?? reveals that the parameter τ plays a critical role in controlling the dispersion and memory-induced damping of the optical beam. Specifically: for $\tau = 0.8$, the beam experiences rapid damping, indicating strong fractional memory and fast energy dispersion. For $\tau = 1.1$, an intermediate regime appears where beam oscillations remain partially coherent over longer distances. For $\tau = 1.6$, the beam maintains higher intensity for a longer distance, reflecting weaker memory effects and reduced damping. These results suggest that the (q,τ)-fractional model provides a flexible tool to describe beam broadening, coherence loss, and dispersion phenomena in fractional optical media. The deformation parameter τ can be tuned to simulate material-dependent response characteristics or engineered optical environments.

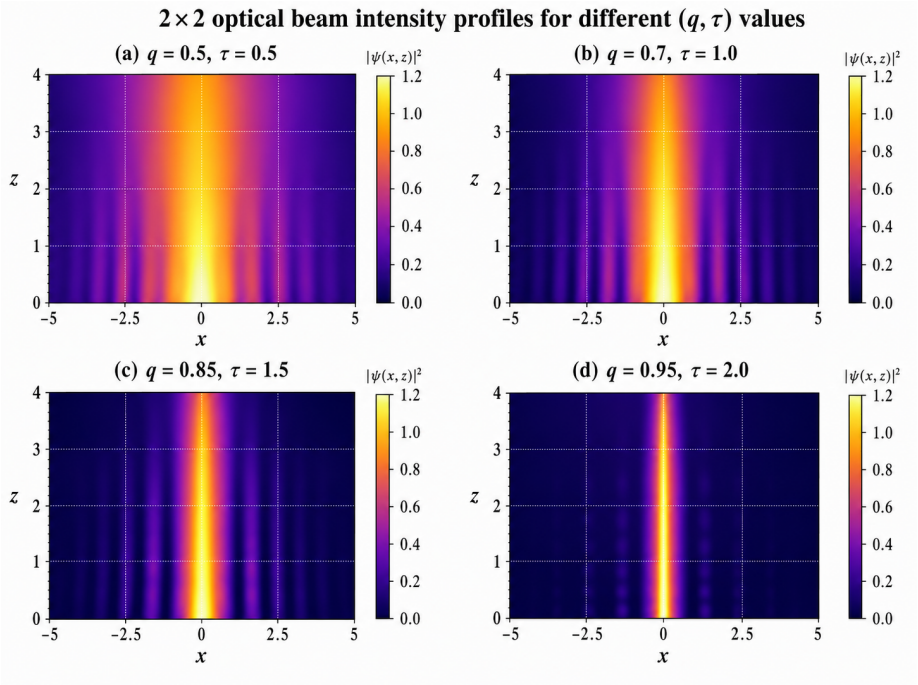


Figure 4. Optical beam intensity profiles $|\psi(x,z)|^2$ for different deformation parameters (q,τ). Panels (a)–(d) correspond to (a) $q = 0.5$, $\tau = 0.5$; (b) $q = 0.7$, $\tau = 1.0$; (c) $q = 0.85$, $\tau = 1.5$; and (d) $q = 0.95$, $\tau = 2.0$, respectively. The results show that increasing q and τ leads to stronger beam localization and reduced spreading along the propagation direction z.

3.4. Physical interpretation and relevance to quantum system modeling

The proposed (q,τ)-framework offers an interpretation of quantum dynamics with discreteness and memory. The parameter q describes lattice-like deformation, while τ adjusts the deformation scale and associated memory weighting. The fractional operators incorporate

weighted past states, and the Hermite-type expansions provide a modal representation of the dynamics. Potential applications include fractional Schrödinger equations, quantum transport, and open quantum systems, subject to verification of the required spectral properties. Table 2

Table 2. Comparison between classical and (q,τ)-fractional methods.

Feature	Classical Method ($\tau = 1$)	(q,τ)-Deformed Method
Memory Kernel	No or limited memory via fixed fractional order	Adjustable memory depth via τ deformation
Discretization Flexibility	Uniform grids with fixed step size	Tunable grid compression using q^{τ} sampling
Orthogonal Basis	Classical polynomials (e.g., Legendre, Chebyshev)	(q,τ)-Bernoulli or (q,τ)-Hermite-type basis
Spectral Accuracy	Limited in non-uniform domains	Higher accuracy in quantum/non-uniform domains
Adaptivity	Less adaptable to complex memory effects	Flexible modeling of scale-invariant and nonlocal dynamics
Quantum Interpretation	Not directly compatible with quantum deformation	Encodes quantum discreteness and time dilation via (q,τ)
Numerical Stability	Sensitive near singularities	Improved handling near singularities via q-grid smoothing

clearly demonstrates the enhanced modeling capabilities introduced by the (q,τ)-deformation. While classical spectral methods operate under fixed discretization schemes and often struggle with memory-driven systems or quantum-scale irregularities, the (q,τ)-framework offers a tunable approach. By adjusting τ, one can emulate systems with deeper memory or long-range correlations, whereas the quantum parameter q introduces granularity suitable for quantum and scale-invariant environments. This is especially useful in fractional PDEs arising in quantum mechanics, anomalous transport, and viscoelasticity. Furthermore, the (q,τ)-Hermite-type polynomials and their corresponding spectral properties yield better localized basis functions, leading to improved convergence for problems with sharp gradients or history-dependent coefficients. The flexibility and generality of this framework suggest it is well-suited to future research in memory-driven quantum models and time-fractional evolution equations.

4. Conclusion

This work investigated a (q,τ)-deformed fractional framework for Schrödinger-type equations using a generalized Gamma function, fractional operators, and Hermite-type polynomial expansions. The analysis established the asymptotic behavior and strict log convexity of the generalized Gamma function and explored modal representations, energy estimates, and solution regularity under the stated assumptions. Numerical illustrations examined how the deformation parameters influence polynomial profiles, modal evolution, and optical beam intensity. The framework provides a basis for further investigations of quantum evolution with deformed spectral structures and memory-dependent dynamics. Future work should clarify the orthogonality conditions of the polynomial family, the convergence and continuation properties of the deformed Mittag–Leffler function, and the compatibility of the fractional operators

with the proposed modal solutions. Further developments may address nonlinear potentials, validated numerical schemes, and physically calibrated applications.

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Data availability

None.

Conflict of interest

There is no conflict of interest to disclose.

Author contributions

Rabha W. Ibrahim: Conceptualization, methodology, mathematical analysis, and writing—original draft. **Dumitru Baleanu:** Validation, supervision, and writing—review and editing. Both authors read and approved the final manuscript.

Declaration of using AI tools

The authors declare that they have not used any type of generative artificial intelligence for the writing of this manuscript, nor for the creation of images, graphics, tables, or their corresponding captions.

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