

RESEARCH ARTICLE

Investigation of stability and oscillatory behavior of allelochemical influence on plants in time delay differential equation model

Mine Babaoglu¹, Dipesh^{2*} and Pankaj Kumar³

¹Department of Mathematics and Science Education, Faculty of Education, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, Türkiye.

²Department of Mathematics, SR University, Warangal-506371, Telangana, India.

³Department of Mathematics, Lovely Professional University, Phagwara, Punjab-144411, India.

*Corresponding Author. Email: dipeshdahal2411@gmail.com (Dipesh)

Article Information

Received: 30 August 2025

Accepted: 1 October 2025

Published: 3 October 2025

AMS 2020 Classification:
34L30

Abstract

The utilizing of bioassay techniques is common for examining how allelochemicals affect plant processes. In general, at higher allelochemical concentrations, processes are impeded, whereas at lower ones, they are boosted. A mathematical model is developed to analyse this kind of reaction. It is assumed that the response of the plant organism is proportional to the allelochemical dose. This plant response is studied for delayed allelochemical dose using time delay differential equation. The feasible non-zero equilibrium point is examined and the stability analysis is studied about the equilibrium point. The system is stable if the delay in allelochemical dose does not cross a fixed critical value. However, the system exhibits limit cycles via Hopf-bifurcation when delay parameter crosses the fixed critical value. Furthermore, in the situation that the diffusion coefficient transpires. Additionally, it exists within the event exhibiting the diffusion coefficient. Analytical results are supported by MATLAB.

Keywords: Allelochemical, plant response, stability, delay parameter, hopf-bifurcation, time delay differential equation.

1. Introduction

Allelopathy refers to a chemical process that is dependent on the release of one or more chemicals into the environment. Allelopathy can be either stimulative or destructive. Allelopathy was described by Molisch [1] as any positive or negative biochemical interaction between microbes and plants of any complexity degree. Plant tissues include stems, foliage, origins, rhizomes, blossoms, fruit, and spores, contain chemicals that have the potential to be allelopathic; nevertheless, leaves and roots are the primary producers of allelochemicals studied by Rice [2]. Numerous processes, such as volatilization, the breakdown of plant wastes, and root exudates, can release allelochemicals into the environment from plants examined by Tukey [3,4], Rice [5], Putnam [6]. Lately, the responses of plants to allelochemicals have been theoretically described by modeling An et al. [7,8], Zhen and Ma [9] through the use of mathematical norms.

For quantitatively measure reactions to chemical compounds at various doses in allelopathy research, bioassay methods are frequently employed. It has been observed that inhibition manifests at high concentrations while stimulation happens at low quantities observed by Lovett [10,11], and Liu and Lovett [12].

Many allelochemical dosage responses are investigated utilising statistical comparison approaches like the Least Significant Difference examined by Paszkowski and Kremer [13], Toro et al. [14], Buta and Spaulding [15], Student's t test, and Duncan's Multiple Range Test by Leather and Einhellig [16] Stevens and Molyneux [17]. However, it may be statistically inefficient to utilize such approaches to analyze the data comprising a sequence of dose rates studied by Dawkins [18]. Nevertheless, the correlation between the allelochemical dosages and the

bioassay organism's responses is not necessarily linear. The reactions of plant growth and development to allelochemicals are primarily nonlinear, similar to many other biological processes.

An et al. [19] theoretically analyzed and modeled the dynamics of allelochemicals released by living plants into the environment, as well as the phytotoxic effects of compounds released from plant residues during decomposition. However, the effect of allelochemicals is not immediate, but takes a maturation time. This period of maturation of allelochemicals is studied Dipesh and Kumar [20-22] by using delay differential equations. Horatmann examined the condition for instability [23]. Pang et al. investigated spatial patterns in the reaction-diffusion model using various diffusion and reaction terms [24]. Paul and Ritson-Williams examined the most current marine chemical ecology literature on benthic cyanobacteria, macroalgae, sponges, octocorals, molluscs, other benthic invertebrates, fish, and seabirds [25]. Tian et al. investigated the spatial flow on allelopathy with cross diffusion [26]. There has been a great deal of effort done to examine the features of solutions to reaction-diffusion systems that describe ecological models [27-29].

The aim of this paper is to investigate the relation of plant response with allelochemicals under what condition the plant response are stable, unstable. In addition, we calculate the wave number of different flow using nonlinear dynamics.

Assumptions:

- The suggested delay differential equation model represents the persistent release of allelochemicals from plant residues or root exudates, therefore reflecting a continuous supply of bioactive substances in the soil.
- The addition of a delayed feedback term accounts for the time lag between allelochemical release, soil modifications, and subsequent reappearance in bioavailable form, which is consistent with documented lagged effects in allelopathic interactions.

2. Mathematical model

Let D a dose of an allelochemical, and T_c the fixed response of untreated control in the bioassay. We may write

$$\frac{dD}{dt} = T_c + f(D). \quad (1)$$

Here $f(D)$ is the effect of allelochemicals. $f(D) > 0$ leads to stimulation and $f(D) < 0$ causes inhibition. Considering $f(D)$ as a simple quadratic equation, we have

$$f(D) = \beta D - \gamma D^2. \quad (2)$$

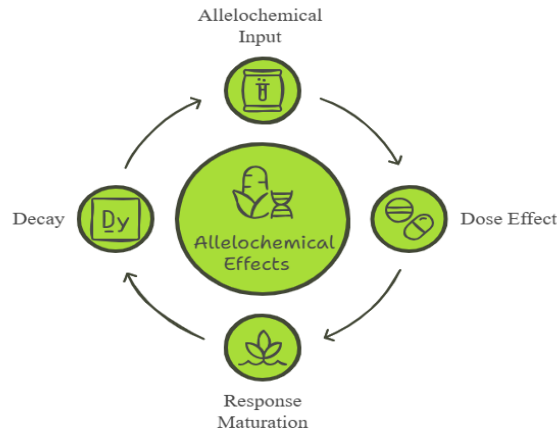


Figure 1. Allelochemical cycle response.

Here β is the rate of input of allelochemicals and γ is the rate of decay of allelochemicals. So, Eq. (1) becomes:

$$\frac{dD}{dt} = T_c + \beta D - \gamma D^2. \quad (3)$$

As the effect of allelochemical dose is not immediate, but takes a maturation time, so a delay parameter τ is introduced in allelochemical dose. The Eq. (3) turns into delay differential equation, and can be finally written as:

$$\frac{dD}{dt} = T_c + \beta D(t - \tau) - \gamma D^2. \quad (4)$$

2.1. Equilibrium point

At the interior non-zero equilibrium point, we have $\beta D(t - \tau) \approx \beta D$

$$\frac{dD^*}{dt} = 0 \Rightarrow D^* = \frac{\beta \pm \sqrt{\beta^2 + 4\gamma T_c}}{2\gamma}.$$

These two values of allelochemical doses are responsible for stimulation and inhibition responses respectively.

2.2. Stability of dose of allelochemicals

Discuss the stability of the response of the plant organism is proportional to the allelochemical dose about the non-zero equilibrium points, then Eq. (4) becomes:

$$\frac{dD^*}{dt} = T_c + \beta D^*(t - \tau) - \gamma D^{*2}. \quad (5)$$

We assume a solution of the form $D^*(t) = e^{\lambda t}$, and differentiating:

$$\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}.$$

Putting the value in Eq. (5), we got:

$$\lambda e^{\lambda t} + \gamma D^{*2} e^{\lambda t} = \beta D^*(t - \tau) e^{\lambda t}. \quad (6)$$

Divide Eq. (6) by $e^{\lambda t}$, it becomes

$$\lambda + \gamma D^{*2} = \beta e^{-\lambda \tau}.$$

The characteristic equation becomes

$$\lambda = \beta e^{-\lambda \tau} - \gamma D^{*2} \quad (7)$$

Let $\lambda = ia$, then Eq. (7) become

$$ia = \beta e^{ia\tau} - \gamma D^{*2}. \quad (8)$$

Using the Euler formula

$$e^{ia\tau} = \cos(a\tau) - i \sin(a\tau)$$

rewrite the Eq. (8), it becomes

$$ia = \beta(\cos(a\tau) - i \sin(a\tau)) - \gamma D^{*2}.$$

separate real and imaginary part:

$$0 = \beta \cos(a\tau) - \gamma D^{*2}, \quad (9)$$

$$a = -\beta \sin(a\tau). \quad (10)$$

From Eq. (9), we have

$$\begin{aligned} \beta \cos(a\tau) &= \gamma D^{*2} \\ \Rightarrow \cos(a\tau) &= \frac{\gamma}{\beta} D^{*2}. \end{aligned} \quad (11)$$

For solution $\left| \frac{\gamma}{\beta} D^{*2} \right| \leq 1$, and from Eq. (10):

$$\frac{a}{\beta} = -\sin(a\tau). \quad (12)$$

Adding the Eqs. (11) and (12), we got:

$$\left(\frac{\gamma}{\beta}D^{*2}\right)^2 + \left(\frac{a}{\beta}\right)^2 = 1,$$

$$a^2 = \beta^2 - \gamma^2D^{*4}.$$

For oscillation solution, we require a to be real

$$\beta^2 > \gamma^2D^{*4}.$$

Case1: If $\beta^2 > \gamma^2D^{*4}$, the oscillation occurs with frequency $a = \sqrt{\beta^2 - \gamma^2D^{*4}}$.

Case2: If $\beta^2 \leq \gamma^2D^{*4}$, oscillations do not occur. If the term within the square root is non-positive, λ remains absolutely real. When λ is negative, the system experiences exponential decay and finally achieves a stable state.

Case3: If $\lambda > 0$, then $D^*(t)$ increase over time and move towards unstable, when $\beta e^{-\lambda\tau} > \gamma D^{*2}$.

For example: When we take the parametric value: $\beta = 2, \gamma = 0.5, D^* = 1$, then

$$\gamma^2D^{*4} = (0.5)^2(1)^4$$

$$\beta^2 = 4.$$

where $\beta^2 > \gamma^2D^{*4}$, oscillation occur, and the frequency is:

$$a = \sqrt{\beta^2 - \gamma^2D^{*4}} \Rightarrow \sqrt{3.75} \approx 1.94$$

Theorem1: Let λ be the root of the characteristic equation $\lambda - \beta e^{-\lambda\tau} + \gamma D^{*2} = 0$,

- 1.1 If $\beta > \gamma$, system shows the oscillatory behavior.
- 1.2 If β is small, the system shows the stability.
- 1.3 There exists a threshold, when delay is greater than the threshold value, the system moves from stability to oscillation (Hopf-bifurcation).

Proof: Let $d(\lambda) = \lambda - \beta e^{-\lambda\tau} + \gamma D^{*2}$, and from Eqs. (9) and (10), solving for τ , and a . We get

$$\tau_c = \frac{1}{a} \arccos\left(\frac{\gamma D^{*2}}{\beta}\right). \quad (13)$$

When delay cross the threshold value, a pair of purely imaginary roots emerges, towards hopf-bifurcation and oscillatory and periodic solution occurs.

2.3. Transversality condition

Differentiate implicitly the characteristic Eq. (7) with respect to τ :

$$\frac{d\lambda}{d\tau} - \beta e^{-\lambda\tau} \left(-\tau \frac{d\lambda}{d\tau} - \lambda\right) = 0$$

$$\frac{d\lambda}{d\tau} + \beta e^{-\lambda\tau} \tau \frac{d\lambda}{d\tau} + \beta e^{-\lambda\tau} \lambda = 0$$

$$\frac{d\lambda}{d\tau} (1 + \beta \tau e^{-\lambda\tau}) = -\lambda \beta e^{-\lambda\tau}$$

$$\frac{d\lambda}{d\tau} = \frac{-\lambda \beta e^{-\lambda\tau}}{1 + \beta \tau e^{-\lambda\tau}}$$

On simplification, we get:

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right) = \frac{\omega^2}{(1+\tau\gamma D^*)^2 + (\tau\omega)^2} > 0.$$

This is always positive. Hence, the transversality condition is satisfied.

3. Numerical example

To validate our theoretical conclusion, we do numerical simulations of the delay differential equation in MATLAB. We examine how different parameters values, namely τ, β , and γ , affect the systems stability and oscillatory dynamics. The values of parameter are $T_c = 0.01, \beta = 0.9, \gamma = 0.0006$.

4. Result and discussion

Figure 2 shows the stability of the behavior of testing organism when there is no dose of allelochemicals and behavior of testing organism remain stable ($\tau = 0$). Figure 3 complements this by presenting a phase portrait under the same no-delay conditions ($\tau = 0$). The phase portrait visually depicts how the system's trajectories evolve over time. Since there is no delay, all trajectories converge smoothly to the equilibrium, confirming that the system behaves predictably when unperturbed by external influences. Figure 4 introduces a delay ($\tau < 1.69999$), which simulates the effect of a dose of allelochemicals on the organism. Initially, the system exhibits noticeable fluctuations—its state variables oscillate before settling down. These fluctuations indicate that the introduction of allelochemicals disturbs the system temporarily. However, the system eventually regains stability, a behavior known as asymptotic stability, where the fluctuations gradually diminish and the system returns to equilibrium over time. Figure 5 further maps out the region in the system's parameter space where stability is maintained when the delay is below the critical value ($\tau < 1.69999$). This figure emphasizes that, within this range, even though there might be some initial disturbances due to the allelochemicals, the overall behavior of the organism remains stable in the long run ($\tau < 1.69999$). Figure 6, the system does not simply return to a stable state; instead, it exhibits sustained periodic oscillations. This transition to periodic behavior is indicative of a Hopf-bifurcation, a type of bifurcation where a fixed point loses stability and gives rise to a limit cycle. In simpler terms, once the delay goes beyond this critical point, the testing organism's behavior switches from a stable state to one with continuous, regular oscillations, highlighting a significant change in the system dynamics ($\tau \geq 1.69999$). Figure 7 represent the log scale view of behavior of testing organism when delay is greater than critical value, i.e. ($\tau \geq 1.69999$).

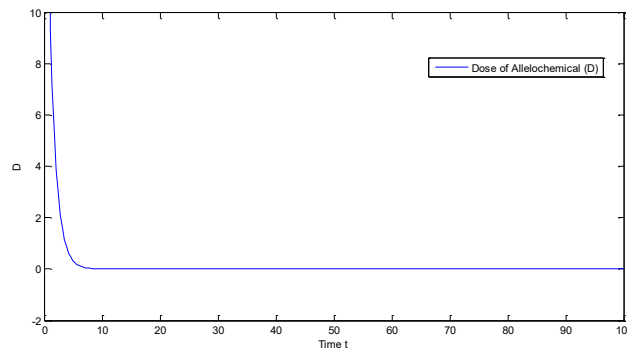


Figure 2. The behavior of the testing organism when there is no delay, i.e., ($\tau = 0$).

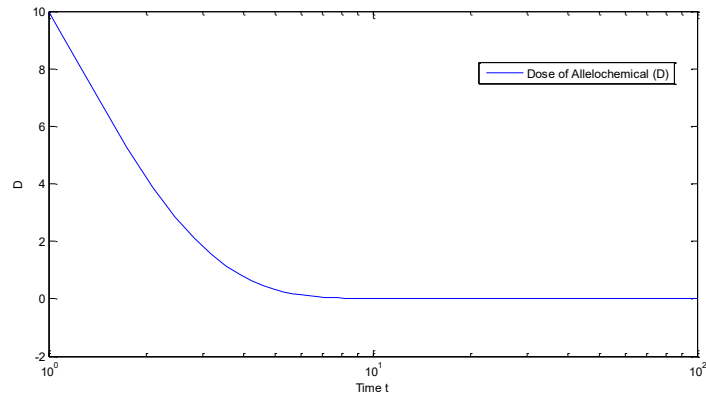


Figure 3. The log phase of behavior of testing organism in the absence of delay, i.e., ($\tau = 0$).

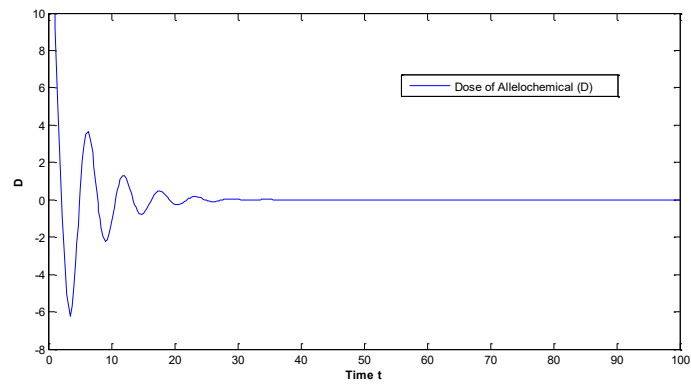


Figure 4. The asymptotic behaviour when delay is less then critical value, i.e. ($\tau < 1.69999$).

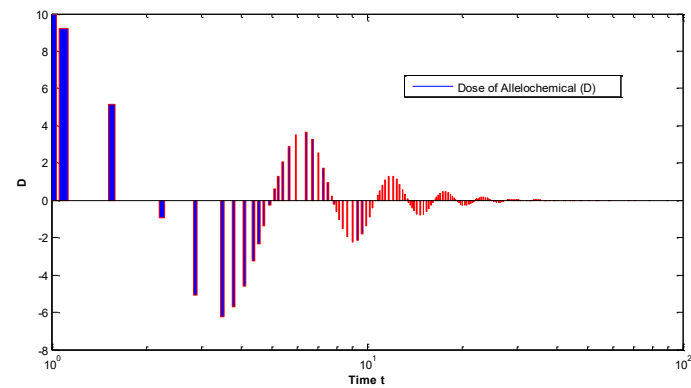


Figure 5. The area of testing organism when delay is less then critical value, i.e. ($\tau < 1.69999$).

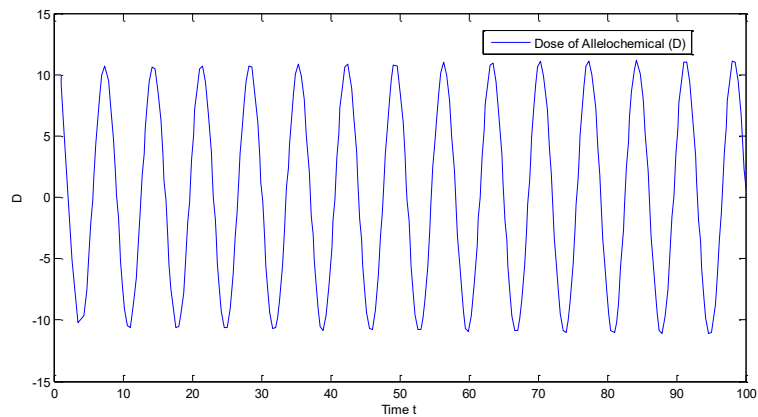


Figure 6. The hopf-bifurcation when delay is greater than critical value, i.e. ($\tau \geq 1.69999$).

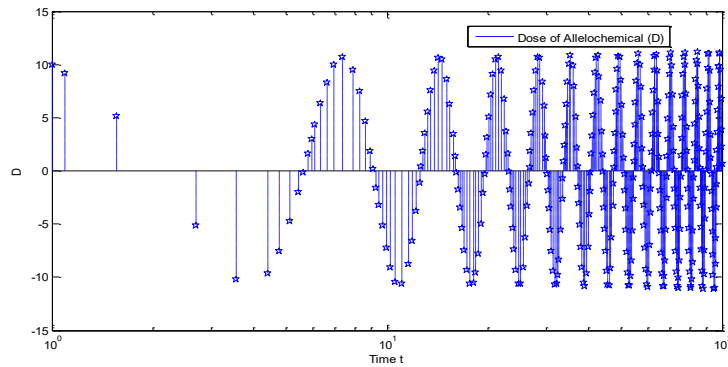


Figure 7. The log phase of behavior of testing organism when delay is greater than critical value, i.e. ($\tau \geq 1.69999$).

5. Conclusion

The current research provides a precise mathematical framework for improved understanding of plant responses to allelochemicals after delayed exposure. According to the findings, all larger amounts hamper plant operations, while smaller quantities aid them. The delay differential equations show that the system remains stable throughout a range of delays about the non-zero equilibrium point. However, a Hopf-bifurcation shown when this limit is considerably surpassed, in addition to the system's transition from total stability to continuous oscillations. A thorough investigation of the stability indicates that the system remains in an ideal state when allelochemicals are absent ($\tau = 0$). Fluctuations occur initially, followed by stabilisation at all values for delays smaller than 1.69999 ($\tau < 1.69999$). When the system reaches a certain threshold ($\tau > 1.69999$), it begins to oscillate. This demonstrates a significant shift in how it operates. This delay mechanism permits allelochemicals to accumulate over time, hence increasing their influence on plant response.

These findings emphasise the relevance of time-dependent aspects in plant-allelochemical interactions, providing insight into ecological modelling and agricultural management while also paving the way for future research on predictive control mechanisms in plant systems.

Acknowledgments

This study was partially presented as an oral presentation on the International Conference on Mathematics and Applied Data Science (ICMADS'25).

Funding

Not Applicable.

Conflict of interest

There is no conflict of interest to disclose.

Author contributions

Mine Babaoglu: Supervision, investigation, mathematical evaluation, formal analysis. **Dipesh:** Writing original manuscript, conceptualization, formal analysis, investigation, methodology, editing and software. **Pankaj Kumar:** Writing original manuscript, and formal analysis, and investigation.

Declaration of using AI tools

The authors declare that they have not used any type of generative artificial intelligence for the writing of this manuscript, nor for the creation of images, graphics, tables, or their corresponding captions.

References

- [1] H. Molisch, "Der Einfluss einer Pflanze auf die andere, Allelopathie". Fischer Jena, 1937.
- [2] E. I. Rice, "Allelopathy". 2nd Edition, Academic Press, New York, 422, 1984.
- [3] H.B. Tukey, "Implications of allelopathy in agricultural plant science". *The Botanical Review*, 35(1), 1969, pp. 1-16.
- [4] H. Jr. Tukey, "The leaching of substances form plants/H. Jr. Tukey". *Annual Review of Plant Physiology*, 1970, pp. 315-324.
- [5] EL. Rice, "Allelopathy". 1st Edition, Academic Press, New York, 343, 1974.
- [6] A.R. Putnam, "Allelopathic research in agriculture: past highlights and potential", 1985.

- [7] M.I.R.J. An, I.R. Johnson, & J.V. Lovett, "Mathematical modeling of allelopathy: biological response to allelochemicals and its interpretation". *Journal of Chemical Ecology*, 19, 1993, pp. 2379-2388.
- [8] M. An, I.R. Johnson, & J.V. Lovett, "Mathematical modelling of allelopathy". I. Phytotoxicity caused by plant residues during decomposition, 1996.
- [9] J. Zhen, & Z. Ma, Z, "Periodic solutions for delay differential equations model of plankton allelopathy". *Computers & Mathematics with Applications*, 44(3-4), 2002, pp. 491-500.
- [10] J.V. Lovett, "The ecological significance of odour in weeds", 1980.
- [11] J.V. Lovett, "Chemicals in plant protection: is there a natural alternative?", 1990.
- [12] D.L. Liu, & J.V. Lovett, "Allelopathy in barley: Potential for biological suppression of weeds", 1990.
- [13] W.L. Paszkowski, & R.J. Kremer, "Biological activity and tentative identification of flavonoid components in velvetleaf (*Abutilon theophrasti* Medik.) seed coats". *Journal of Chemical Ecology*, 14, 1988, pp. 1573-1582.
- [14] G.I. Ramirez Toro, G. R. Leather, & F.A. Einhellig, "Effects of three phenolic compounds on *Lemna gibba* G3". *Journal of Chemical Ecology*, 14, 1988, pp. 845-853.
- [15] J.G. Buta, & D.W. Spaulding, "Allelochemicals in tall fescue-abscisic and phenolic acids". *Journal of Chemical Ecology*, 15, 1989, pp. 1629-1636.
- [16] G.R. Leather, & F.A. Einhellig, "Mechanism of allelopathic action in bioassay". In: Thompson AC (ed), *The Chemistry of Allelopathy, Am Chern Soc Symp Ser 268*, 1985, pp. 197, 205. Washington, DC.
- [17] K.L. Stevens, & R.J. Molyneux, "Castanospermine-a plant growth regulator". *Journal of Chemical Ecology*, 14, 1988, pp. 1467-1473.
- [18] H.C. Dawkins, "Multiple comparisons misused: why so frequently in response-curve studies". *Biometrics*, 39(3), 1983, pp. 789-790.
- [19] M. An, D.L. Liu, I.R. Johnson, & J.V. Lovett, "Mathematical modelling of allelopathy: II. The dynamics of allelochemicals from living plants in the environment". *Ecological Modelling*, 161(1-2), 2003, pp. 53-66.
- [20] Dipesh, & P. Kumar, P., "Investigating the impact of toxicity on plant growth dynamics through the zero of a fifth-degree exponential polynomial: A mathematical model using DDE". *Chaos, Solitons & Fractals*, 171, 2023, pp. 113457.
- [21] Dipesh, & P. Kumar, P., "Modelling the stimulatory and inhibitory allelopathic effects on competing plant populations". In *AIP Conference Proceedings* (Vol. 2435, No. 1). AIP Publishing, 2022.
- [22] Dipesh, & P. Kumar, "A novel approach to 6th-order delay differential equations in toxic plant interactions and soil impact: beyond newton-raphson". *Physica Scripta*, 99(6), 2024, 065236.
- [23] D. Horstmann, "Remarks on some Lotka–Volterra type cross-diffusion models". *Nonlinear Analysis: Real World Applications*, 8(1), 2007, pp. 90-117.
- [24] P.Y. Pang, & M. Wang, "Strategy and stationary pattern in a three-species predator–prey model". *Journal of Differential Equations*, 200(2), 2004, pp. 245-273.
- [25] V.J. Paul, & R. Ritson-Williams, "Marine chemical ecology". *Natural Product Reports*, 25(4), 2008, pp. 662-695.
- [26] C. Tian, L. Zhang, & Z. Lin, "Pattern formation for a model of plankton allelopathy with cross-diffusion". *Journal of the Franklin Institute*, 348(8), 2011, pp. 1947-1964.
- [27] C. Liu, Q. Zhang, X. & Duan, "Dynamical behavior in a harvested differential-algebraic prey–predator model with discrete time delay and stage structure". *Journal of the Franklin Institute*, 346(10), 2009, pp. 1038-1059.
- [28] Z. Liu, S. Zhong, & X. Liu, "Permanence and periodic solutions for an impulsive reaction-diffusion food-chain system with holling type III functional response". *Journal of the Franklin Institute*, 348(2), 2011, pp. 277-299.
- [29] C. Tian, L. Zhang, & Z. Ling, "The stability of a diffusion model of plankton allelopathy with spatio–temporal delays". *Nonlinear Analysis: Real World Applications*, 10(4), 2009, pp. 2036-2046.



All open access articles published in Transactions on Computational Modeling and Intelligent Systems (<http://tcmis.org>) are distributed under the terms of the CC BY-NC 4.0 license (Creative Commons Attribution Non-Commercial 4.0 International Public License as currently displayed at <http://creativecommons.org/licenses/by-nc/4.0/legalcode>) which permits unrestricted use, distribution, and reproduction in any medium, for non-commercial purposes, provided the original work is properly cited.