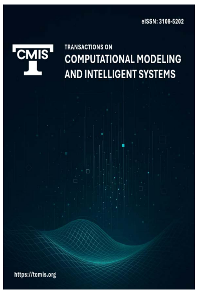

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SYSTEMATIC REVIEW

The logistic models and sigmoid functions: A variety of models and solution perspectives

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Abstract

A systematic review on logistic dynamics models and related sigmoidal functions has been developed, bringing together local and non-local interpretations. Special efforts were applied to show how non-locality can be implemented in logistic models, applying the Volterra approach and the causality concepts. An analysis of solutions of fractionalized logistic models is provided. Special attention is paid to logistic maps, both standard and fractional.

Keywords: Logistic models, sigmoidal curves, non-locality, fractional logistic models.

Contents

1. Introduction

The logistic models of growth are almost 2 centuries old, starting from the seminal article of Verhulst in 1845 [1]. Their fascinating properties allow the modeling of various dynamic processes as well as fitting curves with experimental points in cases beyond the growth problems. Over the course of the last two centuries, numerous improvements and modifications have emerged. Moreover, in the modern era of the 21st century, fractional versions of the classical Verhulst model have been attacked by mathematicians. Commonly, in the literature, no studies are trying to link and compile various aspects of logistic models, starting from their basic definitions, exploring non-local versions, and other applications such as neural dynamics, control, modeling in rheology, material science, etc. One possible explanation of "separated islands of studies" could be found in the overly narrow specializations of people working with various versions of logistic models. This review is an attempt to bring together and compile closely related problems of logistic models, starting from a plethora of non-local versions with modified rate equations, through empirical modifications of the growth curves towards non-local formulations using Volterra's hereditary construction, and attempts to solve the basic logistic model (Verhulst version) in fractionalized versions. Moreover, concepts of how the local logistic models can be transformed to a non-local version and then this technology can be applied in cases when hereditaries are modeled by fractional operators are conceived.

Due to the limitations of the publication format, the focus is only on dynamic models, excluding statistical applications of logistic distributions, and applications in neural network dynamics, as well as modern trends of intelligent systems and artificial intelligence. To some extent, when considering statistical distributions, we can find similar sigmoidal behaviors. However, this is not the task of this study, and for completeness of the exposition, we refer only to two well-known distributions. (see Section 2.2). This introduction is short without exaggerating facts or historical observations because the sequel has sufficient comments, analyses, and thoughts on the topics covered.

In the end, the remarkable diversity of viewpoints, beginning with the straightforward but incredibly effective and powerful Verhulst's model and its "kids," which generated ideas in various directions and applied areas, motivated me to write this systematic review. I hope it will be a helpful source of information, establishing new connections between various fields and stimulating further research in them.

Note: In the following exposition, we try, to some extent, to unify the symbols used. However, often, this is impossible, and we especially mark such cases with comments *in original notations*. Hence, when reading equations, please focus attention on their meaning, but not on the formal use of similar or equal symbols.

2. Preliminaries: Sigmoid functions and logistic distributions

At the beginning we present the basic properties of the sigmoid function that enable further explanations of more complex problems involving it. These properties can be found in many articles and textbooks, but for the sake or clarity and well structuring of the exposition, we have to do that.

2.1. One-dimensional sigmoid function

The one-dimensional sigmoid function (hereafter named simply *sigmoid*) is a solution of the Verhulst equation (see further Eq.(35) in Section 3.2.1), namely

$$y(x) = \sigma(x, w) = \frac{e^{wx}}{1 + e^{wx}} = \frac{1}{1 + e^{-wx}} = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), \quad x \in \mathbb{R} \tag{1}$$

Note: We will use for the one-dimensional case (1) the symbol $\sigma(x)$ in contrast to all other versions of the sigmoidal family (see Section 3.2.1).

This function works in many applications, from population dynamics, where it was originally developed, to activation functions in neural dynamics [2, 3], variable transmuting parameters in transmutation transforms [4], relaxing boundary conditions in applied modeling [5, 6] and approximations in areas such as engineering and material science, where only its shape is the attractive fit to the data without any physical or modeling background (not considered here since we are oriented only towards dynamic models related to sigmoids).

From a pure mathematical point of view, $y(x) = \sigma(x)$ is a solution of the autonomous differential equation

$$\frac{dy}{dx} = y(1 - y), \quad x \in (-\infty, \infty), \quad y(0) = \frac{1}{2} \tag{2}$$

Moreover, we have the following relations satisfied by $\sigma(x)$ (for the sake of simplicity, we assume here $w = 1$)

$$\sigma(x) + \sigma(-x) = 1 \tag{3}$$

$$\lim_{x\to\infty}\sigma(x) = 1, \quad \lim_{x\to-\infty}\sigma(x) = 0, \quad \lim_{x\to 0}\sigma(x) = \frac{1}{2} \tag{4}$$

$$\int \sigma(x)\, dx = \ln(1 + e^{x}) + C \tag{5}$$

Note: The function $\ln(1 + e^{x})$ is known as the Sofplus function [7], and the inequality $e^{x} - \ln(1 + e^{x}) > 0$ holds for all $x \in (-\infty, \infty)$.

Further, it is easy to check that

$$\sigma(x) + \sigma(y) = \sigma(x + y) \tag{6}$$

$$\frac{d}{dx}\left(\frac{d\sigma(x)}{dx}/\sigma^{2}(x)\right) = -e^{-x} < 0 \tag{7}$$

As well as, the Grünbaum-type inequality holds [7, 8]

$$1 + \sigma\left(z^{2}\right) \geq \sigma\left(x^{2}\right) + \sigma\left(y^{2}\right), \quad z^{2} = x^{2} + y^{2}, \quad x, y \in (0, \infty) \tag{8}$$

2.1.1. Derivatives of the sigmoid and its integral

The main attractive feature of the sigmoid is its monotonicity and simple form. Moreover, the form of its derivative makes its application attractive, for example, learning algorithms for back propagation (due to the form of the lower derivatives and its Taylor series expansion). It is what follows in this section that we address the derivatives and the Taylor series expansion.

An interesting property of $\sigma(x, w)$ is that its derivatives concerning x can be written in terms of σ and w only [9], that is (for $x \in (-\infty, \infty)$)

$$\frac{d\sigma(x, w=1)}{dx} = \frac{e^x}{(1+e^x)^2} = \sigma(x)(1-\sigma(x)) \tag{9}$$

$$\frac{d^2\sigma(x, w=1)}{dx^2} = \frac{e^x(1-e^x)}{(1+e^x)^3} = \sigma(x)(1-\sigma(x))(1-2\sigma(x)) \tag{10}$$

$$\frac{d^3\sigma(x, w=1)}{dx^3} = e^x \frac{e^{2x} - 4e^x + 1}{(1+e^x)^4} \tag{11}$$

$$\frac{d\sigma(x)}{dx} = \frac{d\sigma(-x)}{dx} \tag{12}$$

$$\lim_{x\to\pm\infty}\left(\frac{d\sigma(x)}{dx}\right) = 0, \quad \lim_{x\to 0}\left(\frac{d\sigma(x)}{dx}\right) = \frac{1}{4} \tag{13}$$

$$\frac{d}{dx}\left(\frac{\sigma(x)}{x}\right) = \frac{e^{2x}}{x^2(1+e^x)^2} < 0 \tag{14}$$

and its integral were the rate constant w acts as a shifting factor (see Figure 1).

$$\int \sigma(x, w)\, dx = \int \frac{1}{1+e^{-wx}} dx = -\frac{1}{w}\ln\left(\frac{1}{1+e^{wx}}\right) \tag{15}$$

Illustrative plots of $\sigma(x)$, its integral, and its first 3 derivatives are shown in Figure 1

2.1.2. Taylor series expansion

The Taylor series expansions of (1) around x gives

$$\sigma(x+\Delta x) = \sigma(x) + \sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^n\sigma}{dx^n}(\Delta x)^n \tag{16}$$

With the help of (9) and (10) it is possible to establish a recursive procedure for obtaining $\frac{d^n\sigma(x)}{dx^n}$ [9], namely

$$\frac{d^n\sigma(x)}{dx^n} = w^n \frac{d^n\xi}{dx^n}, \quad \frac{d^n\xi}{dx^n} = \sum_{k=1}^{n} (-1)^{k-1} C_k^{(n)} \sigma^k (1-\sigma)^{n+1-k} \tag{17}$$

The coefficients $C_k^{(n)}$ is equal to the Eulerian number $A_{n,k-1}$ from the recurrence

$$A_{r,q} = (q+1)\,A_{r-1,q} + (r-q)\,A_{r-1,q-1} \tag{18}$$

with integer $q, r > 0$, and therefore the Eulerian form of $d^n\xi/dx^n$ is [9])

$$\frac{d^n\xi}{dx^n} = \sum_{k=1}^{n} (-1)^{k-1} A_{n,k-1} \sigma^k (1-\sigma)^{n+1-k} \tag{19}$$

From Eq.(19) it is easy to see that with $n = 1$ we get $\frac{d\xi}{dx} = \sigma(1-\sigma)$ is what we already know. More details and analyzes are available in [9].

Remark 1 (Some versions of the sigmoids for practical use)**.** *Bearing in mind that for $x = 0$ we get $\sigma(0, w) = 1/2$ for any w, then a widely used version in the literature is the activation function [10]*

$$\sigma_{act}(x, w) = \frac{2}{1+e^{-wx}} - 1 \tag{20}$$

Moreover, a degenerate sigmoid function is defined as [11]

$$\sigma_\lambda(x) = \frac{(1+\sigma(x))^{\frac{1}{\lambda}}}{1+(1+\lambda x)^{\frac{1}{\lambda}}} = \frac{1}{1+(1+\lambda x)^{-\frac{1}{\lambda}}} = \frac{1}{2} + \frac{1}{2}\tanh_\lambda\left(\frac{x}{2}\right), \quad \lambda \in (0, \infty) \tag{21}$$

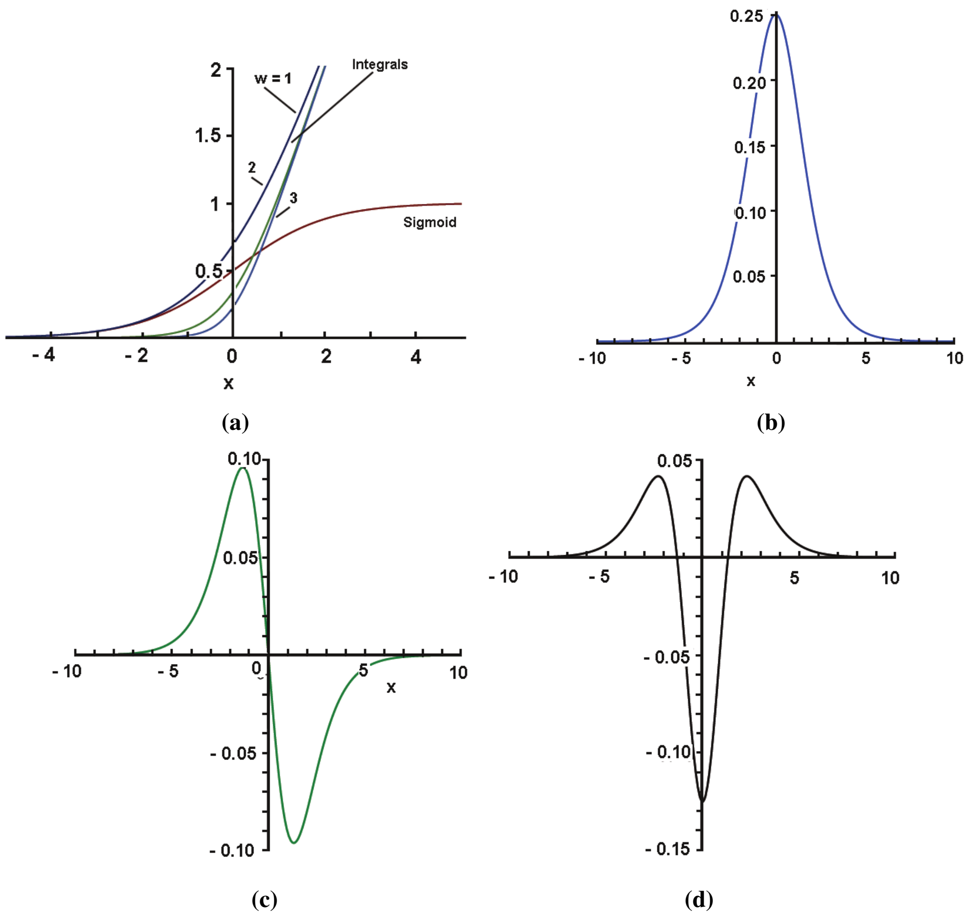


Figure 1. The sigmoid function and its integral with different values of w (a), and its first 3 derivatives: $1st$ derivative (b), $2nd$ derivative (c), and $3rd$ derivative (d), with $w = 1$.

For $\lambda \to 0$ we have $\sigma_x(x) \to \sigma(x)$.
The first derivative is

$$\frac{d\sigma_\lambda(x)}{dx} = \frac{(1+\lambda x)^{-\frac{1}{\lambda}-1}}{\left(1+(1+\lambda x)^{-\frac{1}{\lambda}}\right)^2} > 0, \quad x \in (-\infty, \infty) \tag{22}$$

The degenerate sigmoid obeys relations and inequalities to some extent, resembling those exhibited for the normal (non-degenerate) sigmoid (see more details in [11] since this problem is beyond the scope of this work).

2.2. Logistic and hyperbolic secant distributions

Here we present two distributions that, at least as formulas, resemble the logistic function. We do this especially for the sake of clarity of the exposition and to avoid confusion since they are different in nature and are applied in different fields to model data of different origins (see the Note below).

2.2.1. Logistic Distribution

The name of the logistic function is mainly taken from its cumulative distribution function (CDF) [12]

$$F_{cdf}(x,\mu,s) = \frac{1}{1+e^{-\frac{(x-\mu)}{s}}} = \frac{1}{1+e^{-(x-\mu)t}}, \quad \frac{1}{s} = t > 0 \tag{23}$$

Alternatively, it is called a hyperbolic tangent because [12]

$$F_{cdf-Log}(x,\mu,s) = 1 + \frac{\tanh\left(\frac{x-\mu}{2s}\right)}{2} = 1 + \frac{\tanh\left(\frac{x-\mu}{2}t\right)}{2}, \quad \frac{1}{s} = t > 0 \tag{24}$$

Note: As especially noted by Oliver [13], the logistic distribution should not be confused with the logistic function. The logistic distribution is occasionally used to represent data with patterns similar to the normal distribution because the ogives (the cumulative frequency polygons) are much the same.

The probability density function (PDF) is (hereafter, we prefer to use the variable t instead, s as it is more convenient in the development of further analyses in this article)

$$f_{pdf-Log}(x,\mu,t) = \frac{dF_{cdf-Log}(x)}{dx} = \\ = t\frac{e^{-(x-\mu)t}}{\left(1+e^{-(x-\mu)t}\right)^2} = t\frac{1}{\left(e^{\frac{(x-\mu)}{2}t}+e^{-\frac{(x-\mu)}{2}t}\right)^2} = \frac{t}{4}\operatorname{sech}^2\left(\frac{x-\mu}{2}t\right) \tag{25}$$

In the case when the location parameter μ (coinciding with the mean and the median) is zero, we get

$$F_{cdf-hypS}(x,0,t) = 1+\frac{\tanh\left(\frac{x}{2}t\right)}{2} \tag{26}$$

$$f_{pdf-HypS}(x,0,t) == t\frac{e^{-xt}}{\left(1+e^{-xt}\right)^2} = t\frac{1}{\left(e^{\frac{x}{2}t}+e^{-\frac{x}{2}t}\right)^2} = \frac{t}{4}\operatorname{sech}^2\left(\frac{x}{2}t\right) \tag{27}$$

Plots of PDFs and CDFs of the logistic function for various parameters μ and t are shown in Figure 2.

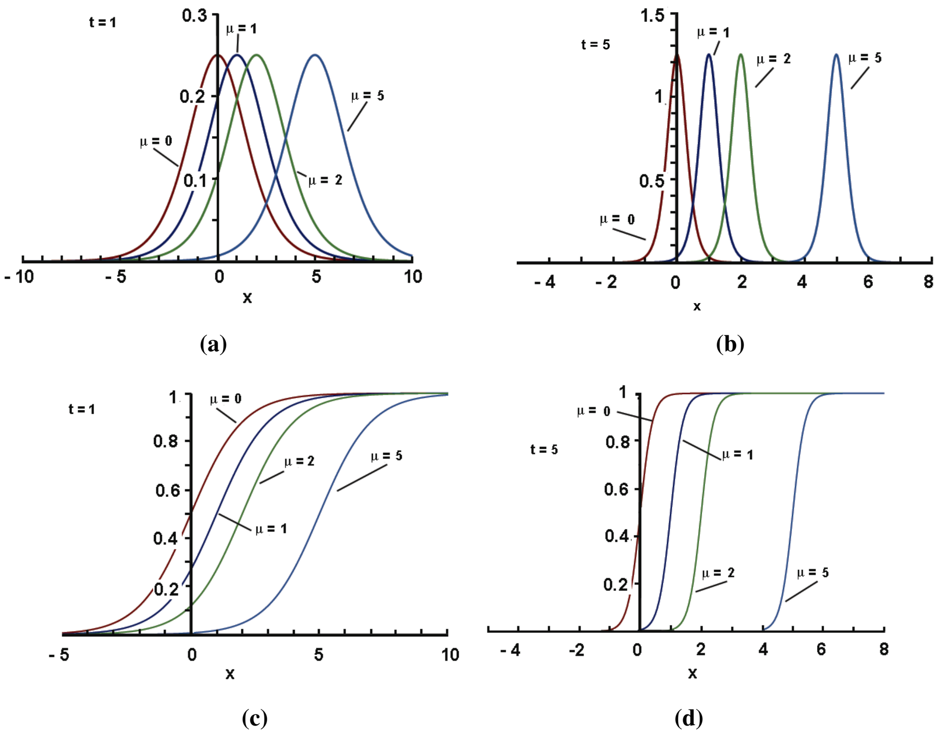


Figure 2. Logistic distributions for two values of t and various values of μ. Top raw-PDFs, Bottom raw-CDFs

2.2.2. Hyperbolic secant distribution

The hyperbolic secant distribution is a continuous probability distribution, as it is seen from the definitions (28) and (29) with PDF and CDF proportional to the hyperbolic secant function [12, 14, 15].

$$f_{pdf-\text{sech}}(x) = \frac{1}{2}\,\text{sech}\,\frac{\pi}{2}x \tag{28}$$

$$F_{cdf-\text{sech}}(x) = \frac{1}{2}+\frac{1}{\pi}\arctan\left(\sinh\frac{\pi}{2}x\right) = \frac{2}{\pi}\arctan\left(\exp\left(\frac{\pi}{2}x\right)\right) \tag{29}$$

Here, both the median and the mean are zero (see Figure 3).

The distribution generalization can include a shifting (location) parameter μ and a scale parameter s

$$f_{pdf-\mathrm{sech}}\left(x,\mu,s\right)=\frac{1}{2s}\,\mathrm{sech}\left(\frac{\pi}{2}\left(\frac{x-\mu}{s}\right)\right) \tag{30}$$

$$F_{cdf-\mathrm{sech}}\left(x,\mu,s\right)=\frac{2}{\pi}\arctan\left(\sinh\left(\frac{\pi}{2}\left(\frac{x-\mu}{s}\right)\right)\right)=\frac{2}{\pi}\arctan\left(\exp\left(\frac{\pi}{2}\left(\frac{x-\mu}{s}\right)\right)\right) \tag{31}$$

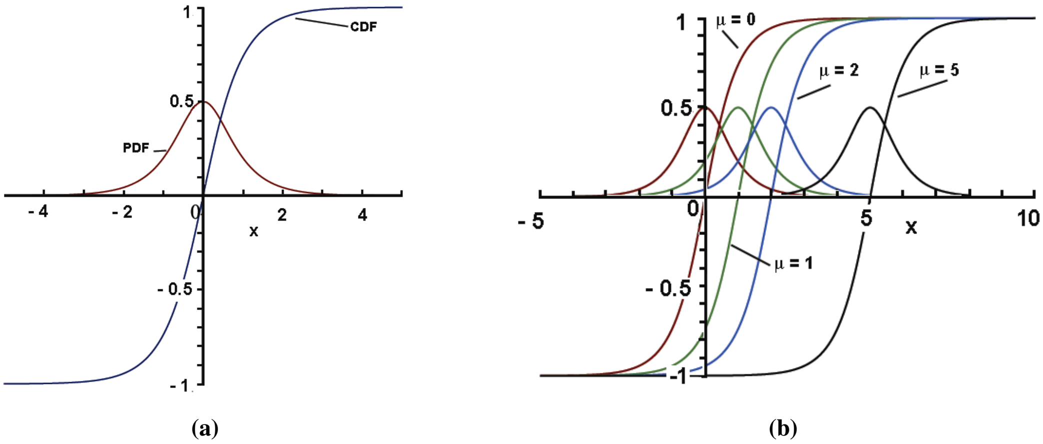


Figure 3. Hyperbolic secant distribution: (a) Non-shifted for $\mu=0$ and (b) Shifted. In all cases $s=1$

Remark 2 (The family of the logistic and hyperbolic secant functions)**.** *The hyperbolic secant distribution and the logistic distribution are special cases of the Chappernowne distribution [16, 17] with CDF and PDF defined as*

$$F_{cdf-Cham}\left(x,\mu,\lambda,x_0\right)=\frac{n}{\cosh\left(\mu\left(x-x_0\right)\right)+\lambda},\quad -\infty<x<\infty \tag{32}$$

$$f_{pdf-Cham}\left(x,\mu,\lambda,x_0\right)=\frac{n}{\frac{1}{2}e^{\mu(x-x_0)}+\frac{1}{2}e^{-\mu(x-x_0)}+\lambda} \tag{33}$$

because $x=\frac{1}{2}\left(e^x+e^{-x}\right)$ *and all parameters* μ, x_0, λ *are positive, while n it is a normalization constant. The Chapernowne distribution has a density* $f_{pdf-Cham}\left(x,\mu,\lambda,x_0\right)$ *that is symmetric concerning the median* x_0 *and exhibits a tail heavier than the normal distribution. For* $\lambda=0$ *,* $\mu=\pi/2$ *and* $x_0=0$ *we get the hyperbolic secant distribution. For* $x_0=0$*,* $\lambda=1$ *and* $\mu=1$ *we get the logistic distribution*

$$f_{pdf-Cham}\left(x,1,1,0\right)=\frac{1}{2+e^x+e^{-x}}=\frac{e^x}{\left(1+e^x\right)^2}=f_{pdf-Log}=\frac{dF_{pdf-Log}}{dx} \tag{34}$$

3. Growth and diffusion modeled by sigmoidal curves: Looking back at things

Many branches of applied research, including biology, economics, epidemiology, and demography, deal with growth and diffusion processes that are, in general, represented by growth curves. Diffusion can be considered a large subset of growth [18]. Since there are no natural phenomena with infinite growth, an upper limit is specified by the nature of the modeled process that finally leads to so-called S-curves, or sigmoidal curves, a term that we will use. The growths could be of two basic types: positive and negative [18]. The most well-known representatives of the positive growths are the logistic [1, 19, 20] and the Gompertz [21] models (see below).

Historically, growth modeling has been associated with some particular populations of humans or animals, virus propagation, cells, marketing, etc.(see more details in the sections developed during the course of this article). The diffusion, however, relies on the spreading of cells, viruses, people, or species in specified areas. That is, the growth is a manifestation of life or an action (process) leading to growing [18].

The main approach in growth curves to model some specific growths is the trend analysis, where fitting some known sigmoidal models is tested against experimental data collected. The methods applied in such tests are different, but the non-linear least squares fitting dominates.

3.1. Growth curve modeling

The growth is considered with the start of a natural process, exhibiting a maximum (maturity) and a final stage (death) [18]. In such processes, the describing functions are commonly referred to as sigmoid curves, but in some cases, they could be exponential curves saturating at high values of the arguments [18].

Considering a growth curve as a measure of how some quantities dominating the process increase over time. Hence, we may consider a dominant parameter or some secondary effects, but finally, we focus on the growth process. A dominant element could be birth, death, cell number, or biomass amount, exhibiting growth in time.

3.2. Dynamic (kinetic) models of growth

3.2.1. The logistic (Verhulst) model

The logistic growth law is a solution of the Verhulst [1] dynamic equation concerning the population size in time

$$\frac{dy(t)}{dt} = my(t) - ny^2(t), \quad y(0) = y_0 > 0 \tag{35}$$

where m and n are constants. Further, Eq.(35) came naturally as a solution to the Lotka model ([19]) when a growth rate is expressed as a series

$$\frac{dy(t)}{dt} = a_1 y + a_2 y^2 + a^3 y^3 + \dots \tag{36}$$

and truncated, for the sake of simplicity, at the quadratic term. The uniqueness of Eq.(35) was tested by Feller [22] and proved that the solutions (population trajectories) allow well to be fitted by either normal distribution curves with time-dependent arguments. Furthermore, Feller suggested a stochastic model describing the process as

$$\frac{dy(t)}{dt} = \left(ay(t) - by^2(t)\right) dt + c\sqrt{y(t)}dW \tag{37}$$

where $y(t)$ is a random population size time, while W denotes a standard Wiener process (all integrals related to it should be considered in the Ito sense). The random growth rate models related to the logistic model will not be further discussed here but a comprehensive analysis is available in [23].

Remark 3 (A non-conventional link of the Verhulst model to reaction-diffusion models)**.** *The Verhulst model can be considered as a diffusionless version of the so-called reaction-diffusion models (Fisher-type models [24, 25], also known as Fisher-Kolmogorov-Petrovsky-Piskunov models [26])*

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + mu^p - mu^{p+q} \tag{38}$$

when the spatial diffusion is neglected, that is the diffusion coefficient D in Eq.(38) is zero and the exponents p and q are set to 1*. As commented by Bacaër in Chapter 20 of [27], the initial step in Fisher's study was the equation $du/dt = au(1-u)$, where we recognize the Verhulst model. The diffusion term was added after that to account for the dispersion randomly in the neighborhood of spatial location x.*

3.2.2. Models of one-stage growths as variants of the logistic models

A more attractive approach that fits well with the context of this article is the modeling through dynamic models producing sigmoidal functions as solutions [18,28,29], resembling, to a greater extent, the solution of the Verhulst model, and we will address next some of them.

That is, considering the dynamic behavior of a certain differentiable variable $x(t)$, the simple kinetic (growth) equation is

$$\frac{dx(t)}{dt} = y(x), \quad t > 0 \tag{39}$$

The function $y(x)$ should have non-negative values throughout its path, thus guaranteeing that the solution $x(t)$ demonstrates a sigmoidal behavior. Moreover, $x(t_2) \geq x(t_1)$, such that $\Delta x = x(t+\Delta t) - x(t) \geq 0$.

An example is the Rayleigh-Nordon (RN) equation [18, 29] (in original notations)

$$\frac{dF}{dt} = at(K - F(t)) \Rightarrow F(t) = K\left(1 - e^{-\frac{a}{2}t^2}\right) \tag{40}$$

As suggested by Berny [18] some factors could be of a power-law that correct the solution in a way that avoids the growth to be proportional to t^2 , that is

$$\frac{dF}{dt} = at^{w-1}\left(K - F\left(t\right)\right) \Rightarrow F\left(t\right) = K\left(1 - e^{-\frac{a}{w}t^{w}}\right), \quad w > 1, \quad t > 0 \tag{41}$$

Some commonly used sigmoidal models, which are variants of the powered logistic model (41), following Berny [18], commented further in this article, are, for instance:

- Logistic (auto-catalytic) model [1, 20]

$$\frac{dy}{dt} = ky\frac{(A-y)}{A} \Rightarrow y = \frac{A}{1+e^{-kt}}, \quad A = \lim_{t\to\infty} y\left(t\right), \quad t > 0 \tag{42}$$

The growth curve is symmetrical around its point of inflexion, and its growth rate declines linearly proportional to x (see plots in Section 2)

- Monomolecular model [20]

$$\frac{dy}{dt} = p\left(A-y\right) \Rightarrow y = A\left(1 - be^{-pt}\right), \quad A = \lim_{t\to\infty} y\left(t\right), \quad t > 0 \tag{43}$$

- Brody's model [28]

$$\frac{dy}{dt} = pe^{-kt} \Rightarrow y(t) = A(1 - Be^{kt}), \quad t > 0, \quad p = ABk = \left(\frac{dy}{dt}\right)_{t=0}, \quad A = lim_{t\to\infty} y(t) \tag{44}$$

- Gompertz's model [20, 21]

$$\frac{dy}{dt} = ky\ln\left(\frac{A}{y}\right) \Rightarrow y = Ae^{-be^{-kt}}, \quad A = \lim_{t\to\infty} y\left(t\right), \quad t > 0 \tag{45}$$

This growth curve, to some extent, resembles the logistic, but it is asymmetrical.

- Richards's model [20]

$$\frac{dy}{dt} = \frac{k}{s}y\left(1 - \left(\frac{y}{s}\right)^{s}\right) \Rightarrow y = \frac{k}{\left(1 + se^{(b-kt)}\right)^{1/s}}, \quad t > 0 \tag{46}$$

- Weibull model [30, 31]

$$\frac{dy}{dt} = sk^{1/s}\ (k-y)\left(\ln\frac{b}{k-y}\right)^{\frac{s-1}{s}} \Rightarrow y = k - be^{kt^{s}}, \quad t > 0 \tag{47}$$

- Morgan-Mercer-Flodin model [32, 33]

$$\frac{dy}{dt} = sk\left(k-b\right)\left(\frac{y-b}{k-b}\right)^{\frac{s-1}{s}}\left(1 - \frac{y-b}{k-b}\right)^{\frac{s-1}{s}} \Rightarrow y = \frac{b + k(pt)^{s}}{1+(pt)^{s}}, \quad t > 0 \tag{48}$$

All parameters k, b, s, p in these growth models are positive constants.

Systematic analyzes of growth curves of the sigmoid family are available in [34–38] and other studies not quoted here. We refer also to the work of Savageau [41] where a general formulation and a survey of special cases are systematically developed.

3.2.3. Attempts to generalizations of the logistic model: The rate equation reconsidered

In most cases when complex systems and derived data have to be modeled, attempts to modify and generalize the logistic curve of growth have been carried out. Here, we will present some examples in this direction.

- Turner et al. model [42]

This model suggests the dynamic equation of growth as

$$\frac{dy}{dt} = \beta x\left(\frac{\phi\left(k\right) - \phi\left(y\right)}{\phi\left(k\right)}\right) \tag{49}$$

with proportional constant (intrinsic growth coefficient as defined in [42]) β to the population y.

The number of available spaces depends on the monotonically increasing function $\phi\left(.\right)$ of the population size [42]. If $\phi\left(.\right)$ it is the identity (i.e., a function mapping any variable into itself) we have $\phi\left(x\right)=x$ and we get the original logistic equation of Verhulst.

Turner et al. [42] suggested a power-law function $\phi(.)^m$ with a positive exponent $m>0$ that led to

$$\frac{dy}{dt}=\beta k^{-m}x\left(k^m-y^m\right) \tag{50}$$

The integration of Eq. (49) yields a solution

$$y=\frac{k}{\left[1+\left(\left(\frac{k}{y_0}\right)^m-1\right)e^{-\beta mt}\right]^{\frac{1}{m}}} \tag{51}$$

A version known as a *generic model* was also developed by Turner et al. [43] where the rate equation is formulated as (in the original notations)

$$\frac{dy}{dt}=\frac{\beta}{k^n}x^{1-np}\left(k^n-y^n\right)^{1+p} \tag{52}$$

and the solution is

$$y=\frac{k}{\left\{1+\left[1+\beta np\left(t-\tau\right)\right]^{-\frac{1}{p}}\right\}^{\frac{1}{n}}} \tag{53}$$

with τ as a constant of integration. In the special case where $p\longrightarrow 0$ the model reduces to the construction of Richards [20]. It was discussed by Nelder [44] and can be expressed as

$$y=\frac{k}{\left[1+e^{-\beta n(t-\tau)}\right]^{\frac{1}{n}}} \tag{54}$$

This is the same as Eq.(56) simply replacing n by $1/m$.

- Nelder's model [44]

Alternatively, Nelder [44] assumed that $\frac{1}{y^m}\left(\frac{dy}{dx}\right)^m=m\frac{1}{y}\frac{dy}{dx}$ and this allowed Eq.(50) to be transformed to the ordinary logistic equation in y^m. With some details, Nelder considered the equation (expressed in the common notations used here)

$$\frac{dy}{dt}=ky\left[1-\left(\frac{y}{A}\right)^{\frac{1}{m}}\right] \tag{55}$$

With $m>0$ the solution to Eq.(55) can be written as

$$y=\frac{A}{\left[1+e^{-\left(\frac{\lambda+kt}{m}\right)}\right]^m} \tag{56}$$

with a constant of integration λ.

Further, A and k are positive because $y\left(t\right)$ increases monotonically in time. From Nelder's solution, it follows that $y^{1/m}$ satisfies the logistic equation, and the upper asymptote (for $t\longrightarrow\infty$) is $A^{1/m}$.

The common problem in logistic law constructions is the definition of the coefficient k, which denotes the maximum of the population size (the plateau of the sigmoid curve). If we assume that the maximum population is a constant over a certain period, then considering a more global case suggests that we have a time-dependent $k\left(t\right)$ upper limit of the population growth; this physically means that the population growth passes through several steps (several saturations) towards the maximum of $k_{\max}=K$. Thus, the time-dependent $k\left(t\right)$ can be expressed as [42]

$$k\left(t\right)=\frac{K}{\left(1+\alpha e^{Bmt}\right)^{\frac{1}{m}}} \tag{57}$$

As a result, the integration of Eq. (50) yields

$$y=K\left[1+\left(\left(\frac{K}{y}\right)^m-1-\frac{\alpha\beta}{\beta-B}\right)e^{-\beta mt}+\frac{\alpha\beta}{\beta-B}e^{-Bmt}\right]^{\frac{1}{m}} \tag{58}$$

For $m = 1$, $\alpha = 0$ (or $B \to \infty$) Eq.(57) reduces to the ordinary logistic growth curve; with $m = 1$ and $B = 0$, the result is also a logistic curve but with modified constants.

- Levenbach's model [45]

Levenbach [45] addressed a generalization of the logistic models using the preceding model of Hotelling [46] where both the relative growth rate $\frac{1}{y}\left(\frac{dy}{dt}\right)$ and y (as a variable, an increasing monotone function) are related as

$$\frac{1}{y}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\ln y\left(t\right) \tag{59}$$

Then, expressing the relative growth rate as a Taylor series about y and truncating to the first order term, we get

$$\frac{1}{y}\left(\frac{dy}{dt}\right) = a + by\left(t\right) \tag{60}$$

and the solution is the ordinary logistic curve.

Further, Levenbach [45] did a step ahead, considering the model (60) as a particular case of the Riccati equation

$$\frac{dy}{dt} = ay + by^2 + c \tag{61}$$

with a solution [45]

$$y\left(t\right) = \frac{\left(d+a\right) - C\left(d-a\right)e^{-dt}}{-2b\left(1+Ce^{-dt}\right)} = \left(\frac{d+a}{d-a}\right)\frac{\left(1-Ce^{-dt}\right)}{-2b\left(1+Ce^{-dt}\right)} \tag{62}$$

were $d = \sqrt{a^2 - 4bc} > 0$, $b < 0$ and C as integration constant.

All solutions expressed as Eq.(62) can be presented as

$$y\left(t\right) = C + \frac{M}{1+Ke^{-dt}}, \quad M = -\frac{d}{b}, \quad C = \frac{d-a}{2b} \tag{63}$$

From the result (63) it follows that the solution is bounded from below by $(d-a)/2b$ and attains a saturation for $-(a+d)/2b$

Since the solution of the Riccati model can start at zero initial state, it is convenient to model the so-called *diffusion growths of things*, such as spreading of telephones, cars, and other stuff in technological forecasting and marketing (see Section 3.3).

- Blumberg's model [47]

The Blumberg model [47] considers a rate equation as

$$\frac{dy}{dt} = ky^a(y_\infty - y)^b \Rightarrow \frac{dy}{dt} = k\frac{y^a}{y_\infty^b}\left(1 - \frac{y}{y_\infty}\right)^b \Rightarrow \int\frac{dy}{y^a(1-y)^b} = ky_\infty^{a+b-1}t + C \tag{64}$$

with the inflexion point at $y_{\inf}/y_\infty = a/(a+b)$ and C as a constant of integration.

For $1 - np = 2 - b$, $\beta = 1$ and $b < 2$ this model is consistent with the generic model of Turner et al. [43]. Also, for $a = b$ we get the Verhulst model.

The Blumberg model has not analytical solutions but the last version (the integral equation) allowed to formulate the function $F\left(b, a; y\right) = \int\frac{dy}{y^a(1-y)^b}$ with $F\left(a, b; y\right) = -F\left(a, b; 1-y\right)$. This allowed to create a set of special cases when analytical solutions are possible (see [47] and the analysis in [38]).

A growth form of the Blumberg type, in a hyperbolic form, represented by a sigmoid curve to a regenerative growth, was developed by Spencer and Coulombe [39] (see also comments of Tsoularis and Wallace in [38]).

$$\frac{dy}{dt} = ry^{\left(1-\frac{1}{n}\right)}\left(1 - \frac{y}{y_\infty}\right)^{\left(1+\frac{1}{n}\right)} \Rightarrow, \quad \frac{y\left(t\right)}{y_\infty} = \frac{(t+a)^n}{b+(t+a)^n}, \quad r = n\left(\frac{y_\infty}{b}\right)^{\frac{1}{n}} \tag{65}$$

where y_∞ is the final value and a, b, n are positive parameters. The inflexion point is $y_{\inf}/y_\infty = (n-1)/2n$.

- Tsoularis-Wallace model [38]

Tsoularis and Wallace [38] formulated a generalized logistic rate equation (autonomous equation)

$$\frac{dy}{dt} = ry^{\alpha}\left[1 - \left(\frac{y}{y_{\infty}}\right)^{\beta}\right]^{\gamma} \tag{66}$$

where α, β, γ are positive constant and $y_{t\to\infty} \to y_{\infty}$ (the carrying capacity). From (66) the relative growth rate attains its maximum at

$$\left(\frac{1}{y}\frac{dy}{dt}\right) = y_{\infty}\left(1 + \frac{\beta\gamma}{\alpha - 1}\right)^{-\frac{1}{\beta}} \tag{67}$$

The change in the variable as $x = (y/y_{\infty})^{\beta}$ allows Eq. (66) to be transformed as

$$\frac{dx}{dt} = \beta r y_{\infty}^{\alpha-1} x^{\frac{(\alpha-1)}{\beta}+1}(1-x)^{\gamma} \tag{68}$$

A detailed solution via separation of variables is available in [38] and we will skip its cumbersome expressions and relevant analyses.

It is important to note that in [38], there are systematic comparative analyses of known logistic models. Some of these models are discussed here, while others are not. This analysis, with many illustrative plots, facilitates a better understanding of the areas where different approaches compete to establish suitable models.

- Logistic model with a polynomial rate [40]

The idea of a polynomial rate function in the Verhulst model, earlier conceived empirically by Pearl and Reed (1922) [55] (see further Eq.(78) in Section 3.2.4) was explored by Kyurkchiev et al. in [40].

The starting model is the reaction network $S + X \overset{k=const.}{\rightarrow} 2X$ modified as

$$S + X \overset{k(t)}{\rightarrow} 2X, \quad k(t) = \sum_{i=0}^{n} b_i t^i \tag{69}$$

For example, the case with $n = 2$ is [40]

$$\begin{cases} \dfrac{ds}{dt} = -\left(b_0 + b_1 t + b_2 t^2\right) sx \\ \dfrac{dx}{dt} = \left(b_0 + b_1 t + b_2 t^2\right) xs \end{cases}, \quad s_0 = s(0), \quad x_0 = x(0) \tag{70}$$

For $s_0 = 1/2$, $x_0 = 1/2$ and with $ds/dt + dx/dt = 0$, the results is $s + x = C_{sx}$, where $C_{sx} = s_0 + x_0$. Then, the model (70) becomes [40]

$$\begin{cases} \dfrac{ds}{dt} = -\left(b_0 + b_1 t + b_2 t^2\right) s(1-s) \\ \dfrac{dx}{dt} = \left(b_0 + b_1 t + b_2 t^2\right) x(1-x) \end{cases} \tag{71}$$

and the solutions are [40]

$$s(t) = \frac{e^{-\left(b_0 t + \frac{b_1}{2}t^2 + \frac{b_3}{3}t^3\right)}}{1 + e^{-\left(b_0 t + \frac{b_1}{2}t^2 + \frac{b_3}{3}t^3\right)}}, \quad x(t) = \frac{1}{1 + e^{-\left(b_0 t + \frac{b_1}{2}t^2 + \frac{b_3}{3}t^3\right)}} \tag{72}$$

In general, for $n > 2$ we have [40]

$$s(t) = \frac{e^{-\left(b_0 t + \frac{b_1}{2}t^2 + \frac{b_3}{3}t^3 + \dots \frac{b_n}{n+1}t^{n+1}\right)}}{1 + e^{-\left(b_0 t + \frac{b_1}{2}t^2 + \frac{b_3}{3}t^3 + \dots \frac{b_n}{n+1}t^{n+1}\right)}}, \quad x(t) = \frac{1}{1 + e^{-\left(b_0 t + \frac{b_1}{2}t^2 + \frac{b_3}{3}t^3 + \dots \frac{b_n}{n+1}t^{n+1}\right)}} \tag{73}$$

3.2.4. Attempts to generalizations of the logistic model: Logistic curve manipulations

- *Two-stage growth (Bi-logistic)*

Meyer [48] considers processes where the carrying capacity, such as in the simple logistic, is not constant but can change as the process evolves, that is, the monitored species at issue expand their niche (habitat). Therefore, there are two processes of logistic growth appearing consequently (as cascades) or in parallel.

This is the so-called Bi-logistic model defined as (in the original notations)

$$N(t) = \frac{k_1}{1+\exp\left[\frac{\ln(81)}{\Delta t_1}(t-t_{m1})\right]} + \frac{k_2}{1+\exp\left[\frac{\ln(81)}{\Delta t_2}(t-t_{m2})\right]} \tag{74}$$

as a superposition of two three-parameter logistic models, where $t_{m1}$and t_{m2} denote the upper times (the points of saturations) of each one-stage logistic sub-model.

A similar approach was used by Tissen et al. [49] with

$$y = \frac{a_1}{1+\exp[-b_1(t-c_1)]} + \frac{f-a_1}{1+\exp[-b_2(t-c_2)]} \tag{75}$$

For more details about the model parameters, meaning, and the asymptotic behavior of this model, we refer to section 2 in [49].

A close approach was applied by Jolicoeur and Pontier [50] with

$$N(t) = \frac{1}{C_1 e^{t/D_1} + C_2 e^{-t/D_2}}, \quad t > 0 \tag{76}$$

where the first exponential term with the rate constant $1/D_1$ corresponds to the population decrease, while the second one, with a rate constant $1/D_2$, measures to the population growth.

For $C_1 = 0$ we get an exponential growth model ; For $D_1 \to \infty$ (too large values) the model approaches the logistic curve. For $C_2 = 0$ we get an exponentially decreasing curve. For $D_2 \to \infty$ we get a logistically decreasing curve. A good analysis and design issues concerning data fitting with Eq. (76) were performed by Fidalgo et al. [51].

The Bi-logistic models can be considered as an attempt to decompose the overall growth process into one-stage (fine-grained) logistic sections following the idea developed in [52] . Modis [52] developed the idea of the analysis of any growth process in terms of sub-processes (cascades). The idea came from the Fourier analysis, the so-called “logistic harmonics”; an old mathematical concept that is frequently used to describe a function as a sum of terms that belong to a particular class of functions.

- Triple-logistic function

As an extension of the idea to create multi-term logistic functions, we refer to Bock and Tissen [53] (in the original notations)

$$y = \alpha\left(\frac{1-p}{1+\exp[-a(t-b)]} + \frac{p}{1+\exp[-c(t-d)]}\right) + \frac{\beta}{1+\exp[-b(t-g)]} \tag{77}$$

where $a, b, c, d, f, g, p, \alpha, \beta$are parameters; $t$is age (the time of individual growth).

An analysis of the accuracy and data fitting problems of Bi-logistic and Triple-logistic was performed by Fujii [54].

- *Pseudo-logistic model with a polynomial approximation of the growth exponents*

Following Pearl and Reed [55] the expression can be consequently transformed as

$$y = \frac{be^{ax}}{1+ce^{ax}} \Rightarrow y = \frac{b}{c+e^{-ax}} \Rightarrow \frac{k}{1+me^{k_a x}}, \quad k = b/c\ , \quad m = 1/c\ , \quad k_a = -a \tag{78}$$

and the dynamic equation of the rate of change dy/dx is

$$\frac{dy}{dx} = -k_a y(k-y) \tag{79}$$

Taking into account that the growth rate dy/dx is dependent on factors that are functions of x (in many cases, the variable x is the time), the factor k_a can be assumed as a function of x, too. Then, denoting, still an unknown, function $f_k(x)$ instead of k_a the rate equation can be expressed as

$$\frac{dy}{y(k-y)} = -f_k(x)\,dx \tag{80}$$

which is a separable equation with a solution

$$y = \frac{k}{1+m^{-kF_k(x)}}, \quad F_k(x) = \int f_k(x)\,dx \tag{81}$$

Then, the principal step is to replace ax in (78) by $F_k(x)$ expressing it as a Taylor series. This step leads to

$$y = \frac{k}{1 + m\exp\left(a_1 x + a_2 x^2 + a_3 x^3 \cdots a_n x^n\right)} \tag{82}$$

For $a_{i>1} = 0$, the model reduces to (78).

The approach increases the number of parameters that should be determined through the data fitting procedure; it was assumed to limit the number of terms in the Taylor series up to three, that is

$$y = \frac{k}{1 + m\exp\left(a_1 x + a_2 x^2 + a_3 x^3\right)} \tag{83}$$

with $a_3 < 0$ in the case of growth and $a_3 > 0$ the case of decay.

As pointed in [55], the relation (83) results in curves resembling in shape the ordinary logistic curve, that is, there are no maximum and minimum points, but only a point of inflexion.

Buis [56] considered and analyzed this purely empirical modification of the logistic model and showed that *posteriori* assumption about the specific growth rate is

$$\frac{1}{y}\frac{dy}{dx} = k(x)(A - y)\left((1 - \frac{y}{A}\right) = c(t)\left(1 - \frac{y}{A}\right) \tag{84}$$

where $c(t) = Ak(t) = -df_k/dt$ is a polynomial with strictly positive real values, at least at $t > 0$.

With the case when $f_k(x)$ is cubic, as result of the sign change of a_3 in (83), we get $c(t)$ increasing and then decreasing: as result the rate equation is then a product of two quadratic polynomials in t and in y, namely

$$\frac{dy}{dt} = (-a_1 + 2a_2 t - 3a_3 t^2\left(1 - \frac{y^2}{A}\right) \tag{85}$$

In addition, there is a left-hand dissymmetry of the growth curve exists for $y < A/2$ [56] that breaks that well-established symmetry of the original logistic model.

In practical applications to fit data from engineering experiments, some versions of the sigmoid functions, or close to them, are widely used. Here we consider versions where the equation of the logistic curve (the CFD in terms of distributions) is modified (see many examples in [57]),among them:

- *Cumulative normal curve* [62, 63]

In the early studies on diffusions, the shape of the cumulative normal curve was widely used because its shape resembles that of the logistic curve (see Figure 7-3 in [63], see also [57])

$$X(t) = a\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{y-\mu}{2\sigma^2}\right)dy + \varepsilon(t) \tag{86}$$

where σ is the dispersion (do not mix it with $\sigma(t)$ used to denote the sigmoid function), and $\varepsilon(t)$ is a stochastic term.

- *Cumulative log-normal curve* [58] (see also [57] for comments)

$$X(t) = a\int_{0}^{t} \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma}}\exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right]dy + \varepsilon(t) \tag{87}$$

This resembles the Gompertz curve and is asymmetric.

In both (86) and (87), for a given μ, the value of σ controls the point of inflection (see, for example, similar effects in the preceding section about the logistic distribution).

- *Log-logistic curve* [59]

The replacement of time t by $\ln(t)$ leads to an asymmetric curve about the inflexion point, namely

$$X(t) = \frac{a}{1 + c\exp(-b\ln(t))} + \varepsilon(t) \tag{88}$$

The inflection point appears earlier than the cumulative sales reach half of the saturation level (the model (88) is related to the diffusion of products discussed in Section 3.3).

- *Non-symmetric responding curve* [60]

$$X(t) = \frac{a}{1 + c\exp\left(-bX^{\delta}t\right)} + \varepsilon(t) \tag{89}$$

where δ is a *non-uniform influencing factor*.

- *Flexible logistic curve (FLOG)* [61]

This is a four-parameter generalization of the logistic curve, and it is general enough to vary the position of the inflexion point between the top and the bottom bounds, as well as being symmetric about it [57]. On its basis, several models have been generated (see [57]), namely:

Inverse power transformation

$$X(t) = \frac{a}{1 + c\exp\left[-b(1+kt)^{\frac{1}{k}} - 1\right]} + \varepsilon(t) \tag{90}$$

Exponential logistic

$$X(t) = \frac{a}{1 + c\exp\left[-b\frac{\exp(kt-1)}{k}\right]} + \varepsilon(t) \tag{91}$$

Box-Cox transformation

$$X(t) = \frac{a}{1 + c\exp\left[-b\frac{(1+t)^{k}-1}{k}\right]} + \varepsilon(t) \tag{92}$$

As commented in [57], the choice of a given model to be applied depends on the log-likelihood, which should be higher than that of the others, and the requirement not to be degenerate.

3.3. Growth models for diffusion of products

Here we address applications of dynamic models that result in equations close to the logistic one, but are not related to biological populations. The common term for such models is *diffusion models*, which refers to the *diffusion (spreading) of technological innovations* or *logistic models for forecasting* [62], popularly known as *Bass models* [64–73]

Remark 4 (Diffusion of products: Rogers' definition [62])**.** *It is noteworthy to mention that the use of the term diffusion should be considered as spreading or increasing the population of something non-biological, such as goods, products, ideas, etc., and there is no spatial diffusions as in the well-known Fick or Fourier models of mass and heat diffusion. Here we provide two definition of Rogers [62] that will clarify the exposition.*

- *Chapter 1 in [62](p.5): Diffusion is the process by which an innovation is communicated through certain channels over time among the members of a social system. It is a special type of communication, in that the messages are concerned with new ideas. Communication is a process in which participants create and share information with one another in order to reach a mutual understanding. This definition implies that communication is a process of convergence (or divergence) as two or more individuals exchange information in order to move toward each other (or apart) in the meanings that they ascribe to certain events.*
- *Chapter 6 in [62] (pp.234-235): The diffusion effect is the cumulatively increasing degree of influence upon an individual to adopt or reject an innovation, resulting from the activation of peer networks about an innovation in a social system. For example, when only 5 percent of the individuals in a system are aware of a new idea, the degree of influence upon an individual to adopt or reject the innovation is quite different from when 95 percent have adopted. In other words, the norms of the system toward the innovation change over time as the diffusion process proceeds, and the new idea is gradually incorporated into the livestreams of the system.*

Note: Following Trajtenberg and Yitzhaki [74], historically, the use of the logistic in diffusion studies is a clear case of methodological *spill-over* from other disciplines, primarily from Biology (e.g., bio-assay) and Population Studies.

The analysis of the Bass model (Section 3.3.2) follows the direct application of the Riccati equation (Section 3.3.1) in the remainder of this section.

3.3.1. The Riccati equation applications

As mentioned by Levenbach [45], the first attempt to apply the Riccati equation to model a logistical growth curve of diffusion of Centrex telephones (a business customer telephone service introduced in 1961) in the 60s of the last century was done by Oliver [13] covering the period from 1961 till 1971.

Oliver's model is a Riccati equation (in the original notations)

$$R_t = \frac{dy\,(t)}{dt} = \alpha + \beta y\,(t) + \frac{\gamma}{y\,(t)} + \varepsilon\,(t) \tag{93}$$

where α, β and γ are empirical coefficients of effects, while $\varepsilon\,(t)$ denotes the statistical error.

The data fitting of Oliver's model to populations of cells [45] (the first application of this model to biological populations) yielded (the source data and the data fitting details are available in Table 3 of [45]) a logistic growth equation

$$y\,(t) = \frac{6452}{1 + 6.90e^{-2.36t}} - 616, \quad t = 1, 2, \ldots 10 \tag{94}$$

Levenbach [45] applied the method of Erkelens [75] and the technique presented previously in Section 3.2.3.

3.3.2. The Bass diffusion model of growth

The Bass model [64, 65] was based on the following assumptions:

- The initial product purchases are made by both *innovators* and *imitators*, and the difference between these two groups is *the buying influence*.
- The importance of innovators will be greater at the beginning but will vanish with time.
- In the following formulations, the coefficients and refer to the *coefficient of innovators* and the *coefficient of imitators*, respectively.

Therefore, the likelihood (the chances) of purchase $f\,(T)$ at time T given that *no purchase has not been made* is

$$\frac{f\,(T)}{1 - F\,(T)} = P\,(T) = p + \frac{q}{mY\,(T)} = p + qF\,(T) \tag{95}$$

with the integral condition $F\,(T) = \int_0^T f\,(t)dt$ and initial condition $F\,(0) = 0$.

The density function $Y\,(T)$ is constructed as an integral of the sales $S\,(t)$ over the interval $t \in [0, T]$, namely

$$Y\,(t) = \int_0^T S\,(t)dt = m\int_0^T f\,(t)dt = mF\,(T) \tag{96}$$

as the total number of product purchases from $t \in [0, T]$.

Then, the sales at $t = T$ are

$$\begin{aligned} S\,(T) = mf\,(T) = P\,(T)\,[m - Y\,(T)] = \left(\left[p + \frac{1}{m}q\int_0^T S\,(t)\,dt\right]\left[m - \int_0^T S\,(t)\,dt\right]\right) = \\ = pm + (q - p)\,Y\,(T) - \frac{q}{m}Y^2\,(T) \end{aligned} \tag{97}$$

The Bass model can be reformulated as [64]

$$\frac{1}{1 - F\,(t)}\frac{dF\,(t)}{dt} = p + qF\,(t), \quad F\,(0) = 0 \tag{98}$$

and expressed as an ordinary differential equation (as a Riccati equation-see the last version in Eq. (99))

$$\frac{dF(t)}{dt} = p\,(1 - F) + qF\,(1 - F) = p - F\,(p - q) - qF^2 \tag{99}$$

The analytical solution is

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}, \quad F(0) = 0 \tag{100}$$

For the parameter mapping, Bass [64] suggested the case $p > q$, leading to a positive peak in time (for $p < q$ the peak is negative). Under this condition, the periodic sale peak is [64]

$$t^* = \frac{1}{p+q}\ln\left(\frac{q}{p}\right) \tag{101}$$

The inflection points are (two inflection points exist) [76]

$$t^{**} = \frac{\ln\left(\frac{q}{p}\right) - \ln\left(2 \pm \sqrt{3}\right)}{p+q} = t^* \pm \frac{\ln\left(2 \pm \sqrt{3}\right)}{p+q} \tag{102}$$

Hence, the time between the inflection point t^{**} and the peak t^* depends only on $(p+q)$ but not on p/q [76].

When the interest in the new products (goods, technologies, ideas,faxes, cellular phones, etc.) is low, the minimal limit (the initial condition) $F_{\min}(t) = F_0$ is assumed to be kept at $F_0 \approx |p|/q$ (if F_0 is higher, then the seed (the initial product release on the market) to the market is larger [76].

A cumulative adoption of the peak in time with seeding was developed by Jain et al. [77], with $F_0 \neq 0$, as

$$F(t) = \frac{1 - \frac{p(1-F_0)}{p+qF_0}e^{-(p+q)t}}{1 + \frac{q(1-F_0)}{p+qF_0}e^{-(p+q)t}} \tag{103}$$

$$t^* = \frac{1}{p+q}\ln\left[\frac{q(1-F_0)}{p+qF_0}\right] \tag{104}$$

With initial seeding on the market, i.e. for $F_0 = 0$ the model and its solutions, reduce to original Bass model. In addition, taking into account that Eq. (102) is valid for $p < 0$, then the condition to have two inflection points is [76]

$$\frac{q(1-F_0)}{p+qF_0} > 2 + \sqrt{3} \tag{105}$$

This condition is completely obeyed when the seed size $F_0 \approx |p|/q$ for any q/p [76] (Orbach summarizes 7 regions of q/p in Table 2 of his work with corresponding boundaries,peak, inflexion points and equilibrium).

The practical meaning of these results is that positive p can be released and at the same time, at least $F_0 \approx |p|/q$ the amount of goods is crucial for starting the market (that is, an initial critical mass is needed to start the product diffusion-see the following Remark 5). Moreover, for $p > |q|$ the product market will attain its 100 % potential in the long range of time (for regular values of $q > 0$). However, for $p < |q|$ the long-range prediction for cumulative sales follows from Eq.(100) as $F(t \to \infty) \to p/|q|$ [76]. Some computer simulations are presented in Figure 4: we can see the low level saturations for cases when $q < 0$ and the vertical shifts for cases with $F_0 \neq 0$.

Remark 5 (The critical mass appearance)**.** *According to Rogers [63], the critical mass occurs at the point at which enough individuals in a system have adopted an innovation so that the innovation's further rate of adoption becomes self-sustaining. For more detailed and expanded explanations, we refer to the 5th edition of the Rogers' book [63].*

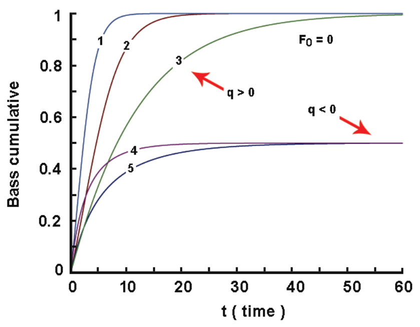
The Bass model presents challenges related to data fitting, primarily due to the application of nonlinear least squares methods (see [78, 79] and the references therein). Although this issue is beyond the scope of this work, we must mention it for the sake of completeness in our exposition.

At the end of this section we refer to an extension of the model (98) as [66]

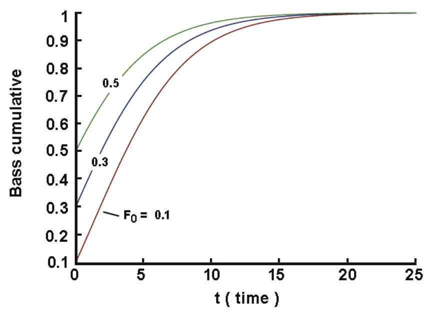
$$\frac{1}{1-F(t)}\frac{dF(t)}{dt} = [p + qF(t)]\,x(t) \tag{106}$$

where the function $x(t)$ relies on percentage change in price and other emerging variables.

In contrast to the analytic solution of the original Bass model, the extended version (106) has no analytical solution and numerical procedures were applied [76].



(a)



(b)

Figure 4. Simulations of the Bass cumulative curve (the parameters p and q are taken from [76], but the plots are created by the author): (a) For $F_0 = 0$: 1-$p = 2, q = 0.4$; 2-$p = 0.1, q = 0.2$; 3-$p = 0.08, q = 0.01$; 4−p=0.1, $q = -0.2$; 5-$p = 0.1, q = -0.2$ and (b) $p = 0.1, q = 0.2$, and $F_0 \neq 0$:(the values of F_0 are chosen arbitrarily, only to show the effect on the separation of the plots.)

4. Some computational and modeling aspects with sigmoidal functions

Since the solutions of the analyzed logistic models revealed numerous sigmoidal functions, it is reasonable to question the computations involved, the related computational techniques, and the approximations that resulted from these applications. In this section, we credit [80–83] and many examples are available in [84]. We try only to show what happens and what can be found beyond the sigmoidal solutions; this is not a complete exposition of the computational problems but only a part of this study contributing to the coherence and more harmonic presentation.

4.1. Approximations based on sigmoidal functions

Now, we focus the attention on approximations based on the sigmoidal function and the computational problems thereof.

4.1.1. The logistic function

- Step (Heaviside) function approximation

The sigmoidal function (1) is a good tool for the approximation of the Heaviside step function $f = ah_0$ by the logistic function $g = s_0$ [80]. The basic problem emerging in such an approximation attempt is the Hausdorff distance, (between two functions f and g) denoted here as H and defined as [80, 85].

$$\rho\left(f, g\right) = \max\left\{\sup_{A\in F(f)} \inf_{B\in F(g)} \|A - B\|, \sup_{B\in F(f)} \inf_{A\in F(g)} \|A - B\|\right\} \tag{107}$$

Then, $H = \rho\left(f, g\right) = \rho\left(ah_0, s_0\right)$ should satisfy the relations $0 < H < a/2$, and $a - s_0\left(H\right)$, such that

$$\frac{a - H}{H} = e^{wH}, \quad 0 < H < a/2 \tag{108}$$

where w is the logistic rate constant.

It follows that for $w \to \infty$ that $H \to 0$. Moreover, the rate constant w as a function of H can be presented as [80]

$$w = w\left(H\right) = \frac{1}{H}\ln\frac{a - H}{H} = O\left(\frac{1}{d}\ln\frac{1}{d}\right) \tag{109}$$

For the sake of simplicity, we assume $a = 1$, so that the rate constant w is the only argument of the logistic sigmoid, which yields $s_0 = 1/\left(1 + e^{-wt}\right)$. Then, the Hausdorff distance H, for $w \geq 2$ is [80]

$$H\left(w\right) = \frac{\ln\left(w + 1\right)}{w + 1}\left[1 + O\left(\varepsilon\left(w\right)\right)\right], \quad \varepsilon\left(w\right) = \frac{\ln\left(\ln\left(w + 1\right)\right)}{\ln\left(w + 1\right)} \tag{110}$$

- Shifted logistic function

When the logistic sigmoid is shifted as $(t-\tau)$ we get the same (i.e., preserving the shape) function, but laterally translated (see, for example, the plots in Figure 2). Since the unit step function also preserves its shape under lateral translation, it follows that the approximation discussed above can be applied ; hence, $s_{w-\tau}(t) = s_0(t-\tau) = \left(1+e^{-w(t-\tau)}\right)^{-1}$, and at the point $-\tau$ we get $s_0(-\tau) == \left(1+e^{w\tau}\right)^{-1}$ [80]. From a computational point of view, the shifted logistic sigmoid is a solution of

$$\frac{dy(t)}{dt} = wy(1-y), \quad y(0) = \frac{1}{1+e^{w\tau}} \tag{111}$$

With an increase in τ, for example, for $w\tau > 30$ the value of the Hausdorff distance is acceptable for the approximation of shifted step functions [80].

- Recurrence generated logistic sigmoids

Consider the recurrence generated sigmoids defined as [81]

$$y_{k+1}(t) = \frac{1}{1+w_{k+1}e^{-w(t+y_k(t))}}, \quad y_{k+1}(0) = \frac{1}{2}, \quad k = 1,2,\ldots \tag{112}$$

From Eq.(112) it follows that $w_{k+1} = e^{w/2}$, $k = 1,2,\ldots$.
In the special case, for example, with one step of recurrence, we have [81]

$$y_1(t) = \frac{1}{1+e^{\frac{w}{2}}e^{-wt}e^{\frac{-w}{1+e^{-wt}}}} \tag{113}$$

In such a case, the Hausdorff distance to the Heaviside function, for $w > e$, should satisfy the condition [81]

$$1-H_1 = y_1(H_1) = \frac{1}{1+e^{\frac{w}{2}}e^{-wH_1}e^{\frac{-w}{1+e^{-wH_1}}}} \tag{114}$$

where H_1 is bounded as [81]

$$H_{1-low} = \frac{1}{\frac{1}{4}\left(w^2+4w+16\right)} < H_1 < \frac{\ln\left[\frac{1}{4}\left(w^2+4w+16\right)\right]}{\frac{1}{4}\left(w^2+4w+16\right)} \tag{115}$$

4.1.2. Blumberg's function

The Blumberg model (64), termed also as *hyper-log-logistic*, can be represented as [82]

$$\frac{dy}{dt} = ky^{1-\frac{1}{\beta}}(1-y)^{1+\frac{1}{\beta}} \tag{116}$$

This is the same as the formulation used by Tsoularis in [38], for the Spencer-Coulombe model [39] (see Eq.(65)). That is, regarding Eq.(64), the exponents are replaced as $a = 1-1/\beta$ and $b = 1+1/\beta$; for $\beta \to \infty$ we get $a = b = 1$ and Eq.(116) (the Verhulst model) . However, for $\beta = 1$, Eq.(116) is a second-order kinetic equation [82]

$$\frac{dy}{dt} = k(1-y)^2 \tag{117}$$

Hereafter, in this section, we preserve the notation used in [82] (i.e., we will use β instead n in Eq.(65).
The solution of Eq.(116) suggests that $y(t)$ is defined by the nonlinear equation [82]

$$\left(\frac{y}{1-y}\right)^{\frac{1}{\beta}} = 1+\frac{kt}{\beta} \tag{118}$$

Differentiation of both sides of Eq.(118) and a rearrangement yields Eq.(116), which confirms Eq.(118) as its solution. Further, Eq. (118) can be represented as [82]

$$y(t) = 1-\frac{1}{1+\left(1+\frac{kt}{\beta}\right)^{\beta}}, \quad y(0) = \frac{1}{2} \tag{119}$$

Now, concerning the approximation of the Heaviside function, the bounds of the Hausdorff distance to the function (119) is [82].

$$\frac{1}{2+k} < H < \frac{1}{1+\sqrt{k}} \tag{120}$$

The one-sided distance of the Blumberg function to the Heaviside function reduces to that established for the Verhulst model, because both models merge as $\beta \to \infty$ [82], which is obvious, as commented above. However, this is a special case, since as we saw the Blumberg model has no analytical solution but only some tabulated cases of an integral function.

The function (119) allows generation of a recurrence family [82]

$$y_{j+1}(t) = 1 - \frac{1}{1 + \left[1 + \frac{k}{\beta}\left(t - \frac{1}{2} - y_j(t)\right)\right]^{\beta}}, \quad j = 1, 2, \ldots \tag{121}$$

Remark 6. *Computational tasks of sigmoidal functions and their use to approximate other functions envisage a careful measure of the approximation accuracy, and as demonstrated above, the Hausdorff distance is an adequate approach to do that. We presented only a few examples, but the approach can be extended towards other sigmoidal functions; we refer to the quoted works and especially to [84], where detailed developed problems and pertinent computer codes are available.*

5. Non-local approach to logistic models

Now, we start with logistic models with heredity in their historical chain of development. This step is important and germane in light of modern trends to apply fractional (non-local) operators to what is already established in practice as models. Traces of this approach could be detected back to the 60s-70s of the last century [86–89, 91–96, 98–100] (see the comprehensive analysis of Frenkel and Choudhury [101]) and continued in the 21st century [102] .

The heredity effects incorporated in the mathematical models alter the formulations, since as a result, we get integro-differential equations [86, 87]. In this context, the most well-described examples of this approach exist in [86, 99] where the hereditary effects on the logistic models are systematically presented, and we will follow this style as a highly needed step towards the next level, implementing fractional operators.

5.1. The main idea towards the non-locality implementation

5.2. Non-local models based on Volterra equation of second kind

For a single-species the model with heredity, following Davis [86], as well as Frenkel and Choudhury [101], can be formulated as

$$\frac{dy}{dt} = ay - by^2 + \lambda y \int_{s=0}^{t} K(t,s)\, y(s) ds, \quad t \geq 0 \tag{122}$$

The population is assumed to be continuous and nonnegative in time. As in the Verhulst model, the parameters a and b are assumed positive, but λ can be either positive or negative. This model has a positive initial condition $\lim_{t\to 0} y(t) = y_0$.

The kernel $K(t,s)$ is the correlation function (memory), and the entire model is a Volterra non-linear integro-differential equation. The choice of the kernel $K(t,s)$ depends on the physical background of the modeled populations and is a matter of argument when the model has to be constructed. For example, in [101] it was decided, for the sake of simplicity, to perform the analysis assuming $K(t,s) = 1$, which was done in [86]

5.2.1. Transformation into a non-linear differential equation

The introduction of new variables [101] such as , $y(t) = (a/b)\,\Theta(t)$, $t = \xi/a$ and $\Lambda = \lambda/(a/b)$, allows Eq. (122) to transformed as

$$\frac{d\Theta}{d\xi}(\xi) = \Theta(\xi)\left[1 - \Theta(\xi)\right] + \Lambda\Theta(\xi) \int_{\xi_0=0}^{\xi} \Theta(\xi_0)\, d\xi_0, \quad \xi \geq 0 \tag{123}$$

with the initial condition $\lim_{t\to 0}\Theta(\xi) = \Theta_0$, and $\Theta(\xi) \in C^{\infty}(\mathbb{R})$, $\xi \in [0,\infty)$ and $\Theta_0 \in (0,1)$;Λ is a coefficient.

Now, the first idea coming to mind is to convert Eq.(123) into an equivalent non-linear differential equation. The first attempts in this direction [86, 99] (see also [101]) are by a direct differentiation and then eliminating the remaining integral term, which finally increases the order of the equation in $\Theta(\xi)$, namely

$$\Theta(\xi)\frac{d^2\Theta(\xi)}{d\xi^2}-\left(\frac{d\Theta(\xi)}{d\xi}\right)^2-\Theta^2(\xi)\frac{d\Theta(\xi)}{d\xi}+\Lambda\Theta^3(\xi) \qquad (124)$$

The increase in the order of the new equation to the second order in time invokes two auxiliary conditions to be imposed: the first is $\lim_{t\to 0}\Theta(\xi)=\Theta_0$, but the second one is $\lim_{t\to 0}\frac{d\Theta(\xi)}{d\xi}=\Theta_0(1-\Theta_0)$ (obtained by setting $\xi=\xi_0$ in (123). This new formulation poses the problem posed (i.e., it is not ill-posed) [101, 104].

5.2.2. Transformation into an equivalent integral equation

The integration of Eq. (123) directly (with the condition $\lim_{t\to 0}\Theta(\xi)=\Theta_0$) with respect to ξ [87, 101], met computational difficulties. Alternatively, if we present Eq.(123) as [101]

$$\frac{d}{d\xi}\ln\Theta(\xi)=1-\Theta(\xi)+\Lambda\int_{\xi_0=0}^{\xi}\Theta(\xi_0)\,d\xi_0,\quad \lim_{t\to 0}\Theta(\xi)=\Theta_0,\quad 0\le\xi\le\xi_{\max} \qquad (125)$$

and integrating the result is

$$\Theta(\xi)=\Theta_0 e^{\xi}\exp\left[\int_{\xi_0=0}^{\xi}K(\xi,\xi_0)\,\Theta(\xi_0)\,d\xi_0\right],\quad 0\le\xi\le\xi_{\max} \qquad (126)$$

The kernels is $K(\xi,\xi_0)=-1+\Lambda(\xi-\xi_0)$ and it is non-singular [101]; not additional analyzes concerning the type of the kernel exist in [101].

5.2.3. Transformation into an alternative differential equation

Following Frenkel and Choudhury [101], the transformed model (124) , which is a hyperbolic integro-differential equation, is hard to analyze. Because of that, they suggest the function (as a cumulative population up to a current time ξ)

$$C(\xi)=\int_{\xi_0=0}^{\xi}\Theta(\xi_0)d\xi_0 \qquad (127)$$

Then, Eq. (123), with $\lim_{t\to 0}\Theta(\xi)=\Theta_0$, can be presented as

$$\frac{d^2C}{d\xi^2}=\frac{dC}{d\xi}\left(1-\frac{dC}{d\xi}\right)+\Lambda C(\xi)\frac{dC}{d\xi},\quad \xi\ge 0 \qquad (128)$$

with $lim_{\xi\to 0}C(\xi)=0$ and $lim_{\xi\to 0}\frac{dC(\xi)}{d\xi}=\Theta_0$

Frenkel and Choudhury [101] developed a detailed analysis of Eq.(128) with both analytical (through $\Theta(\xi)$ expansion as a series) and numerical (by a conventional Runge-Kutta method) solutions, and for a better understanding of the outcomes of this study, we suggest a thorough reading of all its sections.

5.3. Other non-local formulations of the logistic model

We now address other approaches to implement non-locality in the logistic model. To a greater extent, the models considered in this section are directly postulated (that is, there are no derivations from first principles, which can be easily explained by the tasks: mathematical analyses and solutions rather than model build-ups).

5.3.1. Cushing's studies

Cushing [93] considered the integro-differential equation

$$\frac{dN}{dt} = \lambda N(t)\left[1 - \frac{1}{c}\int_0^{\infty} N(t-s)\, d\alpha(s)\right], \quad t \in (-\infty, \infty) \tag{129}$$

modeling a single-species population growth when the rate of growth depends on (is affected by) the population size at previous times. The integral term is interpreted as the accumulated or hereditary effects of past population sizes. This is the model introduced by Volterra [103]; λ is a coefficient .

The integral term can be considered as a Stieltjes integral (due to the original formulation $d\alpha(s) = f(s)\, ds$), which can be easily transformed into a Riemann integral. That is, when $\alpha(s)$ represents a unit step function ($s = 0$) there are no memory effects, and we get the Verhulst logistic equation.

As commented by Cushing [93], Miller in the above mentioned work [87] considered the case when

$$\alpha(s) = bu_s(s) + \int_0^{s} f(\eta)\, d\eta, \quad f(s) \geq 0, \quad \int_0^{\infty} f(s) ds < \infty \tag{130}$$

which is equivalent to $d\alpha(s) = f(s)\, ds$.

From (130) it follows that $b > \int_0^{s} f(s)\, ds$ [87, 93]. As a result, all solutions approach c for long times ($t \to \infty$).

On the other hand, with $\alpha(s)$ as a unit step function and $s = \tau$ (delay) we get a constant time-lag formulation of the logistic model.

$$\frac{dN}{dt} = \lambda N(t)\left[1 - \frac{1}{c} N(t-\tau)\right] \tag{131}$$

The basic assumption considered by Cushing in [93] is

$$\int_0^{\infty} d\alpha(s) = 1, \quad s \in [0, \infty) \tag{132}$$

and it supports the Volterra hypothesis $d\alpha(s) = f(s)\, ds$, $f(s) > 0$, and $\int_0^{\infty} f(s) ds = 1$.

Further, if $N(t)$ is bounded (the main feature of the logistic model) and continuously differentiable with bounded derivative, and ω-periodic, then $\int_0^{\infty} N(t-s) d\alpha(s)$, and the differentiation can be taken under the integral [93] .

The further studies of Cushing [95, 96] considered cases when the net birth rate in the logistic model (the coefficient m in the ordinary Verhulst model (Eq.(35)) is a periodic function of time (as a result of biological or seasonal fluctuations). We will consider the final results of [95, 96] because this direction is outside the main line of the present work.

5.3.2. Brauers's analysis

Brauer [92] considered the Volterra equation (in the original notations)

$$x(t) = f(t) + \int_0^{t} g\left[x(x-s)\right] P(s)\, ds \tag{133}$$

In Eq. (133) $g(x)$ is the number of the members added to the population in unit time when the population size is x; that is ratio $g(x)/x$ is the relative rate of growth per unit time. The function $P(s)$ is the probability that a member of the population survives to age t. Hence, $g\left[x(x-s)\right] P(s)\, \Delta s$ and there is a change in the population from time $(t-s)$ to another $(t-s+\Delta s)$ with members surviving to age s at the time t.

Brauer [92,94] was interested in the boundedness of the solution (a fact we know about the ordinary logistic equation), assuming that $f(t)$ is nonnegative, continuous, and of bounded variation on $0 \leq t < \infty$, such that $f(\infty) = \lim_{t\to\infty} f(t)$. Also, $P(s)$ is nonnegative, nonincreasing, and differentiable on $0 \leq t < \infty$ obeying the condition $\int_0^\infty P(s)ds$, and is normalized, so that $P(0) = 1$.

With $g(0) = 0$ the assumption that $g(x)$ is continuous and nonnegative on $0 \leq x < \infty$ results that the limit of the solution c should satisfy the equation [92]

$$c = f(\infty) + g(c) \int_0^\infty P(s)ds \tag{134}$$

as a good support of this result, if the linear integral equation [92, 97] is considered

$$z(t) = F(t) + \int_0^{t-\tau} z(t-s)a(s)\,ds, \quad t \geq \tau \tag{135}$$

as well as $z(t), F(t)$ $(F(0) = 0)$ and $a(t)$ are continuous and differentiable on $0 \leq t < \infty$; also $a(t)$ is bounded variation such that $\int_0^\infty a(t)dt < \infty$.

Then, if $\int_0^\infty a(t)dt = 1$ and $\int_0^\infty ta(t)dt = m_1 < \infty$, as well as $\int_0^\infty t^2 a(t)dt < \infty$ with $\int_0^\infty F(t)dt = b < \infty$ we have a unique solution of Eq. (135), for $\tau \leq t < \infty$, which is nonnegative such that $\lim_{t\to\infty} z(t) = b/m_1$ yielding that

$$\lim_{t\to\infty} z(t)\, e^{-\sigma(t-\tau)} = \frac{\int_\tau^\infty e^{-\sigma(t-\tau)} F(t)\,dt}{\int_\tau^\infty te^{-\sigma(t-\tau)} a(t)\,dt} \tag{136}$$

Brauer demonstrated that with a change in the variable $t - \tau$ and defining $\upsilon(u) = z(u+\tau) = z(t)$ $\Phi(u) = F(u+\tau) = F(t)$, for $u > 0$ it is possible to represent Eq. (133) as

$$\upsilon(u) = \Phi(u) + \int_0^u z(t-s)a(s)\,ds = \Phi(0) + \int_0^u \upsilon(t-s)a(s)\,ds, \quad t \geq \tau, \quad u \geq 0 \tag{137}$$

which, to a greater extent, resembles the constructions we will consider further when fractional models are discussed.

5.3.3. Zwanzig's derivation of a non-local (generalized) Verhulst law

After the preceding analyzes, where properties of the modeling equations and their solution were briefly outlined, we would like to stress on the technology of model derivation. In this direction, we credit Zwanzig [89], whose model-building approach could be considered as a good template towards the development of new non-local logistic equations.

The Zwanzig derivation starts with a set of Volterra equations, raising the following question:

- What are the conditions under which the species' evolution would follow the Verhulst model ?

In answering this formulated problem, and considering a population of n species, Zwanzig suggested that at the time $t = 0$ all species (from 1 to n) with $N_j(0)$ are close to their equilibrium level Q_j, that is

$$N_j(0) \simeq Q_j, \quad j = 1, 2, \ldots n \tag{138}$$

However, the zeroth species may also be significantly out of equilibrium [89] and therefore a situation with $N_0(0) \neq Q_0$ is possible. Then, the rate of the zeroth species' evolution beyond the equilibrium is

$$\frac{dN_0(t)}{dt} = N_0(t) \sum_{j=1}^{n} a_{0j}\left(N_j(t) - Q_j\right) \tag{139}$$

The evolution of a certain j species ($j \neq 0$) can be represented as a sum of two terms

$$\frac{dN_j(t)}{dt} = N_j(t)\sum_{k=1}^{n} a_{jk}(N_k(t) - Q_k) + N_j(t)\, a_{j0}(N_0(t) - Q_0) \tag{140}$$

The linearization applied considered the deviation $N_j - Q_j$ (for $j \neq 0$) by introducing a new variable x_j so that $N_j = Q_j + \sqrt{Q_j} x_j(t)$. This generates a new matrix $c_{jk} = \sqrt{Q_j} a_{jk} \sqrt{Q_k}$, where $j, k \neq 0$. After these steps Eq.(140) can be presented as

$$\frac{dx_j(t)}{dt} = \left(1 + \frac{x_j}{\sqrt{Q_j}}\right)\sum_{k}^{n} c_{jk}x_k + \sqrt{Q_j}\left(1 + \frac{x_j}{\sqrt{Q_j}}\right) a_{j0}(N_0 - Q_0) \tag{141}$$

Zwanzig's hypothesis is that at the initial state, all values of x_j are negligibly small, and we can assume that $x_j(t) \ll \sqrt{Q_j}$, which are equivalent to $\log(Q_0/N_0(0)\ ) \ll Q_1/2Q_0$. With this step Eq.(141) can be linearized as [89]

$$\frac{dx_j(t)}{dt} = \sum_{k}^{n} c_{jk}x_k + \sqrt{Q_j} a_{j0}(N_0 - Q_0) \tag{142}$$

Considering Eq.(142) as an initial value problem, Zwanzig's solution involved an exponential matrix operator $U_{jk} = \exp(ct)_{jk}$. Skipping details, the equation about $dN_0(t)/dt$ becomes [89]

$$\frac{dN_0(t)}{dt} = -N_0(t)\int_0^t K(t-s)\left[N_0(s) - Q_0\right] ds + N_0(t)\, F(t) \tag{143}$$

The kernel $K(t)$ represents the hereditary effects on the population and is explicitly defined as [89]

$$K(t) = -\sum_{j}\sum_{k} a_{0j}\sqrt{Q_j} U_{jk}(t)\sqrt{Q_k} a_{k0} \tag{144}$$

where the noise $F(t)$ is completely determined by all initial deviations of $x_k(0)$ as

$$F(t) = \sum_{j}\sum_{k} a_{0j}\sqrt{Q_j} U_{jk}(t)\, x_k(0) \tag{145}$$

As intermediate outcomes, the Zwanzig derivation of a hereditary population rate equation is based on a semi-linearization of the Volterra model, from which it starts. Further, when N_0 is enough smaller than the equilibrium value Q_0 we have $\log[Q_0/N_0(0)\ ] \ll Q_1/2Q_0$. However, if $N_0(0)$ is much larger than Q_0 we have and $N_0(0)/Q \ \ll Q_1/2Q_0$.

- Zwanzig's assumptions about the heredity kernel

The hereditary kernel is determined by all interacting constants a_{jk} (see for clarity of this statement Eq.(144) and read the explanation in Remark 7). As commented by Zwanzig, the matrix c_{jk} is antisymmetric and if n is even, then its eigenvalue spectrum is a set of complex conjugate pairs $\pm i\omega_m$(the value of m is within the range $[1, n/2\ ]$), but when n is odd, there is an additional eigenvalue [89]. Zwanzig decided that the kernel should be

$$K(t) = \sum K_m \cos(\omega_m t), \quad K(0) > 0 \tag{146}$$

An example provided by Zwanzig, is the case with the exponential kernel $K(t) = K(0)\exp(-t/T\ )$, where T is the relaxation time (in the original Zwanzig notation). Then, neglecting the noise, i.e. $F(t) = 0$, the generalized Verhulst equation, in Zwanzig's interpretation, becomes

$$\frac{dN_0(t)}{dt} = -K(0)\, N_0(t)\int_0^t e^{-\left(\frac{t-s}{T}\right)}\left[N_0(s) - Q_0\right] \tag{147}$$

As is it is well-known, and Zwanzig demonstrated that, by a change in variables $w(t) = \log[N_0(t)/Q_0\ ]$ we get

$$\frac{dw(t)}{dt} = -K(0)\, Q_0\int_0^t e^{-\left(\frac{t-s}{T}\right)}\left[w(s) - 1\right] \tag{148}$$

Differentiating both sides of Eq. (148) concerning t , we obtain [89]

$$\frac{d^2 w(t)}{dt^2} = -\frac{1}{T}\frac{dw(t)}{dt} - K(0)\,Q_0\left[e^w - 1\right], \quad w(t=0) = \log\left[N_0(0)/Q_0\right], \quad \frac{dw(t=0)}{dt} = 0 \quad (149)$$

Note: Just from a formal point of view, Eq.(149) is close to Eq.(124) and Eq.(128), concerning the second order derivatives in the left-hand sides.

Remark 7 (The appearance of the convolution in the Zwanzig model)**.** *Following Widder [90], the convolution of two series $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ is defined by a new series $\{c_n\}_0^\infty$, where*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a_{n-k} b_k \quad (150)$$

This happens when the two series are multiplied together , i.e. $\sum_0^\infty a_k z^k \sum_0^\infty b_k z^k = \sum_0^\infty c_k z^k$. And, the convolution of two-sided series

$$c_n = \sum_{-\infty}^{\infty} a_k b_{n-k} = \sum_{-\infty}^{\infty} a_{n-k} b_k \quad (151)$$

presents itself when two Laurent series are multiplied [90].
As especially mentioned by Widder, ***hardly less familiar is the continuous analogue*** *of (151)*

$$c(x) = \int_{-\infty}^{\infty} a(x-s)b(s)\,ds = \int_{-\infty}^{\infty} a(s)b(x-s)\,ds \quad (152)$$

If $a(x)$ and $b(x)$, both vanish on $(-\infty, 0)$, then Eq.(151) reduces to [90]

$$c(x) = \int_{0}^{\infty} a(x-s)b(s)\,ds = \int_{0}^{\infty} a(s)b(x-s)\,ds \quad (153)$$

as a continuous analog of (150).

Remark 8 (Zwanzig's interpretation of Eq.(149))**.** *Zwanzig interpreted Eq.(149) as an equation of motion of a particle, where the coordinate is w, the mass is T, and the potential is defined as $U(w) = K_0 Q_0 T\left[e^w - 1\right]$; the friction contributes with the term $-dw(t)/dt$. From this point of view, if the relaxation time T is infinite, or we have a constant $K(t)$, the motion is periodic. However, with large T, the motion oscillates but is damped. Moreover, as mentioned by Zwanzig, the defined integral of $K(t)$ yields $k = K(0)\,T$, a constant corresponding to the rate constant in the Verhulst logistic curve. Further, last but not least, in the case with $T \to 0$ (no relaxation of the motion process exists, we get directly the Verhulst model (in the light of the following discussion on non-locality in the logistic model, the Verhulst model is local in time, that is obvious by the use of local derivative dy/dt and no relaxation suggested in it derivation).*

6. Constructions of hereditary logistic models with fractional operators

In light of the general approach to constructing hereditary models, we emphasize applications of fractional operators and the main principles on which it is based. Developing the concept, we envisage two principal approaches:

- Volterra integral equations with memory kernels known from the fractional calculus
- Causality principle, taking into account the dynamic character of the logistic models

6.1. Fractionalization based on the Volterra equation of second kind

To develop this concept, let us go back to Eq.(122), rewriting it with a memory integral (a convolution term)

$$\frac{dy}{dt} = ay - by^2 + \lambda \int_0^t R(t-s)y(s)\,ds \quad (154)$$

Since there is no restriction, from a mathematical point of view, we may consider that the memory kernel is singular and of Abel's type $R(t) = t^{1-\alpha}/\Gamma(\alpha)$. Then, we may construct a hereditary logistic equation in a Volterra manner as

$$\frac{dy}{dt} = ay - by^2 + \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} y(s)\, ds \Rightarrow \frac{dy}{dt} = ay - by^2 + \lambda^{RL} I_t^{\alpha} y(t) \tag{155}$$

where ${}^{Rl}I_t^{\alpha}[y(t);s] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} y(s)\, ds$, $0 < \alpha < 1$ is the Riemann-Liouville fractional integral.

Note: From the construction of eq.(155) it follows that when the second (convolution) term vanishes (since $R(t) \to 0$ for $t \to \infty$), we get the local logistic model with its asymptote $y(t) \to 1$ as $t \to \infty$.

Equation (155) can be transformed by differentiating on both sides for t, which yields

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(ay - by^2\right) + \lambda \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s)\, ds \tag{156}$$

and we got an equation in terms of the Riemann-Liouville fractional derivative (the last damping term in Eq.(156)).

$${}^{RL}D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s)\, ds \tag{157}$$

Further, rearrangements in (156) result in

$$\frac{d^2y}{dt^2} = \left(a\frac{dy}{dt} - 2by\frac{dy}{dt}\right) + \lambda^{RL} D_t^{\alpha} y(t) \tag{158}$$

Now, we can see that we got almost the same equation as (128) by Frenkel and Chodhury [101], where the coefficients $a = b = 1$, but now the last term is a convolution integral (a fractional derivative).

Moreover, bearing in mind that at zero initial condition, i.e., with $y(0) = y_0 = 0$, under the circumstances of the population models considered here, the Riemann-Liouville (158) and the Caputo derivative (160) are equivalent, then, for the sake of completeness of the exposition, we may write Eq.(158) as

$$\frac{d^2y}{dt^2} = \left(a\frac{dy}{dt} - 2by\frac{dy}{dt}\right) + \lambda^{C} D_t^{\alpha} y(t) \tag{159}$$

$${}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dy(s)}{dt} ds \tag{160}$$

Developing the technology to construct hereditary equations of Volterra type involving fractional operators, we may generalize it by setting the construction of the rate equation as

$$\frac{dy}{dt} = ay - by^2 + \lambda^{*} I_t^{\alpha} y(t) \tag{161}$$

where ${}^{RL*}I_t^{\alpha} y(t)$ is a convolution (constitutive fractional integral of Riemann-Liouville type [105]). The kernel considered should satisfy simultaneously the conditions to be used as memory and to be suitable to model the relaxation of the considered dynamic process (see examples of such constructions in [106] and [107].

Differentiating both sides of Eq.(161) we get a fractional differential equation

$$\frac{d^2y}{dt^2} = \left(a\frac{dy}{dt} - 2by\frac{dy}{dt}\right) + \lambda^{RL*} D_t^{\alpha} y(t) \tag{162}$$

That is, starting with a convolution integral as that in Eq.(154), we always obtain hyperbolic equations with damping terms expressed as fractional derivatives of a Riemann-Liouville type.

As a generalization of the above concept, if the growth model is of an autonomous type and local (of integer order), then the steps towards the hereditary construction of Volterra type involving fractional operators (a generalized definition not related only the power-law memory kernel)

$$\frac{dy\left(t\right)}{dt}=F\left(y\left(t\right)\right)\Rightarrow hereditary\Rightarrow\frac{dy\left(t\right)}{dt}=F\left(y\left(t\right)\right)+\lambda^{RL*}I_{t}^{\alpha}y\left(y\right) \tag{163}$$

$$\frac{d^{2}y\left(t\right)}{dt^{2}}=\frac{d}{dt}F\left(y\left(t\right)\right)\Rightarrow hereditary\Rightarrow\frac{d^{2}y\left(t\right)}{dt^{2}}=\frac{d}{dt}F\left(y\left(t\right)\right)+\lambda^{RL*}D_{t}^{\alpha}y\left(y\right) \tag{164}$$

With this, we close this section, and it might be expected that future studies will accept this concept in model building, which is thermodynamically consistent and corresponds to the causality principle [106, 107]. The simple function $F\left(y\left(t\right)\right)=ay-by^{2}$ is only an example, but any other, as shown in Sections 3.2.2 and 3.2.3 could be considered.

6.2. Fractionalization based on the Volterra equation of the first kind

If the hereditary model is based on the Volterra equation of the first kind, then the general construction is

$$\frac{dy(t)}{dt}=\lambda\int_{0}^{t}R(t-s)\left[ay(s)-by^{2}(s)\right]ds \tag{165}$$

Then, we can see that assuming a kernel of Abel type, $R(t)=t^{1-\alpha}/\Gamma(\alpha)$ we get

$$\frac{dy(t)}{dt}=\lambda\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{1-\alpha}\left[ay(s)-by^{2}(s)\right]ds \tag{166}$$

where the right-hand side is the Riemann-Liouville fractional integral.

Differentiating both sides with respect to t, we obtain

$$\frac{d^{2}y(t)}{dt^{2}}=\lambda\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{0}^{t}(t-s)^{-\alpha}\left[ay(s)-by^{2}(s)\right]ds\Rightarrow\frac{d^{2}y(t)}{dt^{2}}=\lambda^{RL}D_{t}^{\alpha}\left[ay(s)-by^{2}(s)\right] \tag{167}$$

In general, there is no restriction, from a mathematical point of view, to use a kernel different from the power-law, and then using Eq.(165), we can obtain

$$\frac{d^{2}y(t)}{dt^{2}}=\lambda\frac{d}{dt}\int_{0}^{t}R(t;\alpha)\left[ay(s)-by^{2}(s)\right]ds\Rightarrow\frac{d^{2}y(t)}{dt^{2}}=^{RL*}D_{t}^{\alpha}\left[ay(s)-by^{2}(s)\right] \tag{168}$$

where ${}^{RL*}D_{t}^{\alpha}$ is a generalized form of a Riemann-Liouville derivative ireespective of the memory kernel. These steps are one of the possible transformations of Eq.(165) into an equivalent differential equation, and we will see in the next Section 6.4 how this can be done in a different way using the causality principle.

6.3. A fractionalization of the Volterra logistic model with an integral kernel: A good example

A very good example supporting the general scheme of fractionalization presented above was developed by Monteiro et al. [108] based on the logistic equation in the Volterra convolution form [103] (see Eq.(129))

$$\frac{dy\left(t\right)}{dt}=ry\left(t\right)\left(1-\int\limits_{0}^{t}R\left(t-s\right)y\left(s\right)ds\right) \tag{169}$$

The model assumes that the carrying capacity is delayed or is affected by the past [108]. Delays in the population dynamics can be due to many reasons, such as not enough food, diseases, insufficient stimuli, etc., and it can be performed at a time scale different from that of the relaxation process (concerning the relaxation of the memory kernel) [109].

The kernel is considered the generalized Gamma Mittag-Leffler probability distribution [110]

$$f\left(x\right)=\begin{cases}0, & elsewhere\\ Cx^{-\beta}e^{ax}E_{\alpha,\beta}\left(-\lambda x^{\alpha}\right), & x\geq 0\end{cases} \tag{170}$$

where $C=a^{\beta}+a^{\beta-\alpha}\lambda$ is the normalizing factor; The Mittag-Leffler distribution does not have finite moments [108, 112] .

The following hypotheses have to be satisfied:

- $a=0,\quad\lambda>0,\quad 0<\alpha=\beta\leq 1$

- $-a^{\alpha} < \lambda \leq 0, \quad a > 0, \quad \alpha, \beta > 0$
- $0 \leq a^{\alpha}, \quad a > 0, \quad 0 < \alpha \leq 1, \quad \alpha \leq \beta$

Now, with $R(t) = f(t)$ Eq. (169) takes the form [108, 112]

$$\frac{dy(t)}{dt} = ry(t)\left(1 - \int_0^t C(t-s)^{\beta-1} e^{-a(t-s)} E_{\alpha,\beta}\left(-\lambda(t-s)^{\alpha}\right) y(s)\, ds\right) \tag{171}$$

Note: In such a case, we have the so-called integral kernel [108, 113].

Then, using the fractional Riemann-Liouville integral and derivative, as well as applying the Laplace transform, precisely that

$$\int_0^{\infty} Ct^{\beta-1} e^{-at} E_{\alpha,\beta}\left(-\lambda t^{\alpha}\right) dt = C\frac{a^{\alpha-\beta}}{a^{\alpha}+\lambda}, \quad 0 \leq \lambda < a^{\alpha}, \quad a > 0, \quad 0 < \alpha < 1, \quad \alpha \leq \beta \tag{172}$$

because, upon such conditions, $E_{\alpha,\beta}\left(-\lambda t^{\alpha}\right)$ is positive and completely monotonic [111].

Further, the logistic model can be written as [108, 112]

$$\frac{dy(t)}{dt} = ry(t) - ry(t)Ce^{-at}\mathcal{L}^{-1}\left\{\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}\left(-\lambda t^{\alpha}\right)\right]\mathcal{L}\left[\left(e^{at}\right)y(t)\right]\right\} \tag{173}$$

Next, in terms of the Riemann-Liouville fractional integral, the result is

$$\frac{dy(t)}{dt} = ry(t) - ry(t)Ce^{-at}D_t^{1-\beta}\left[e^{at}W\right], \quad W = \int_0^t E_{\alpha}\left(-\lambda(t-s)^{\alpha}\right)e^{-a(t-s)}y(s)\, ds \tag{174}$$

With the help of the Leibniz rule, and that $E_{\alpha,\alpha}(x) = \sum\limits_{n=0}^{\infty} \frac{n!}{\Gamma(\alpha n+1)}\frac{x^n}{n!} = \alpha E^2_{\alpha,1+\alpha}(x)$ we have [108, 112]

$$\begin{aligned}\frac{dW}{dt} = y(t) - \int_0^t &\left\{\lambda(t-s)^{\alpha-1}E_{\alpha,\alpha}\left[-\lambda(t-s)^{\alpha}\right]\right\} + \\ &+ aE_{\alpha}\left[-\lambda(t-s)^{\alpha}\right]e^{-a(t-s)}y(s)\, ds\end{aligned} \tag{175}$$

Compactly, the model can be presented as [108, 112]

$$\begin{cases}\dfrac{dy(t)}{dt} = \left\{ry(t)\left[1 - Ce^{-at}D^{1-\beta}\left(e^{at}W\right)\right]\right\}, \\ \dfrac{dW}{dt} = y(t) - \lambda e^{-at}D^{1-\alpha}\left(e^{at}W\right) - aW\end{cases} \tag{176}$$

It should be noted that for $\alpha > 1$ it followed $D^{1-\alpha}f(x) = I_t^{\alpha-1}f(x)$ (the fractional Riemann-Liouville integral of order $\alpha - 1$) . Also, we have the same for $\beta > 0$. In terms of the Caputo derivative, instead of the fractional Riemann-Liouville derivative, it follows that $W(0) = 0$ (a logical result since both derivatives are identical at zero initial conditions).

In [108, 112] the authors especially stress the attention to the fact that the fractionalization process does not affect the time derivative $dy(t)/dt$ in contrast to the surrogate fraction mode commented in Section 7 where it is replaced mechanistically by a certain fractional derivative. This is in agreement with the general concept of implementing the non-locality in the logistic model, as we saw in all examples developed earlier. The idea of implementing the Gamma Mittag-Leffler function was also developed in [112, 114].

6.3.1. Some special cases

From the model developed, we may obtain some special cases :

- Mittag-Leffler distributed delay:

With $0 < \alpha = \beta < 1$ and $a = 0$, we have a fractional Poisson process, where the kernel is defined as the probability density function of the Mittag-Leffler distribution (alos termed as Mittag-Leffler PDF kernel).

Then, the model becomes

$$\frac{dy(t)}{dt} = ry(t)\,1 - \int_0^t \lambda (t-s)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda (t-s)^{\alpha}\right] y(s)\,ds \tag{177}$$

or

$$\begin{aligned} \frac{dy(t)}{dt} &= ry(t)\left(1 - \lambda D^{1-\alpha} W\right) \\ \frac{dW}{dt} &= y(t) - \lambda D^{1-\alpha} W \end{aligned} \tag{178}$$

For $\alpha \to$ the model to recover an exponentially distributed delay.

- Erlang's probability distribution kernel :

Defining the probability density function of the Gamma distribution as a delay kernel when $\lambda = 0$ in Eq.(170), one obtains [115]

$$\frac{dy(t)}{dt} = ry(t)\,1 - \int_0^t \frac{a^{\beta}}{\Gamma(\beta)} (t-s)^{\beta-1} e^{-a(t-s)} y(s)\,ds, \quad \beta > 0 \tag{179}$$

If $\beta = n \in \mathbb{N}$, that is, we have the Erlang distribution. For $\beta = 1$ in Eq. (179) the following function can be defined

$$w(t) = \int_0^t a e^{-a(t-s)} y(s) ds \tag{180}$$

i.e, the memory kernel is ae^{-at} .

Since $\frac{dw(t)}{dt} = ay(t) - aw(t) = \frac{y(t)-w(t)}{\mu}$, with μ denoting the mean of the distribution, Eq.(179) is equivalent to the following system

$$\begin{cases} \dfrac{dy(t)}{dt} = ry(t)\left(1 - w(t)\right) \\ \dfrac{dw(t)}{dt} = \dfrac{y(t) - w(t)}{\mu} \end{cases} \tag{181}$$

- Gamma distributed delay:

With $\lambda = 0$ the model (179) reduces to

$$\begin{aligned} \frac{dy(t)}{dt} &= ry(t)\left[1 - Ce^{-at} D_t^{1-\beta}\left(e^{at} W\right)\right] \\ \frac{dW(t)}{dt} &= y(t) - aW(t) \end{aligned} \tag{182}$$

The model (182) lost Mittag-Leffler elements but remains fractional as long as $\beta \notin \mathbb{N}$ (as long as the kernel is Erlang's PDF) [108]

6.4. Fractionalization based on the causality principle

Now, consider the growth model as a dynamic system where the relation between the input and output, the causality principle [116], can be expressed as a convolution integral

$$y_{out}(t) = \int_{-\infty}^{\infty} R(t,s) y_{in}(s)\,ds \Rightarrow y_{out}(t) = \int_0^{\infty} R(t,s) y_{in}(s)\,ds \tag{183}$$

where $R(t)$ is a function of influence (memory function) relating the moment value of the function $y(t)$to its past; the recent moments have stronger effects than the distant ones, since the function $R(t)$ should decay and vanish as $t \to \infty$

In the second form of Eq.(183), the lower terminal is set to zero, because in any population, in general, any dynamic growing process, *the time is not the chronological time*, *but the intrinsic time* (the time of process duration), starting at the point $t = 0$, because there is no growth before that [117, 118] ; the same principle, considering the memory kernel, needs $R(t)$ to be causal function, that is $R(t < 0) = 0$.

Hence, considering the dynamic growth models, in a local form, as

$$\underbrace{\frac{dy(t)}{dt}}_{output} = \underbrace{F(y(t))}_{input} \tag{184}$$

we may construct a causal version as (this is a Volterra equation of first kind, as commented earlier)

$$\frac{dy}{dt} = \int_0^t R(t-s;\alpha) F(y(s))ds \Rightarrow \frac{dy}{dt} = {}^{*}I_t^{\alpha} F(y(t)), \quad 0<\alpha<1 \tag{185}$$

The right-hand side of Eq.(185), presented as ${}^{*}I_t^{\alpha} F(y(t))$, can be considered as *a general construction of a memory integral*, where the kernel $R(t)$ could be of any type involved in the known fractional operators (we discussed this above).

Now, as in the preceding examples, let us consider that ${}^{*}I_t^{\alpha} F(y(t))$ is of a Riemann-Liouville type, such that

$$\frac{dy}{dt} = {}^{RL}I_t^{\alpha} F(y(t)) \Rightarrow \frac{dy}{dt} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} F(y(s))ds \tag{186}$$

or more compactly as

$$\frac{dy}{dt} = {}^{RL}I_t^{\alpha} F(y(t)) \Rightarrow \frac{dy}{dt} = {}^{C}D_t^{1-\alpha} F(y(t)), \quad t>0, \quad 0<\alpha<1 \tag{187}$$

For the Caputo derivative and the fractional integral, we have: ${}^{C}D_t^{\alpha} I_t^{\alpha} f(t) = {}^{C}D_t^{\alpha} {}^{C}D_t^{-\alpha} f(t) = f(t)$ and this allows us to transform Eq.(187) by applying to both sides the operator ${}^{C}D_t^{\alpha-1}$ (it is the left inverse to the fractional integral) , that is

$${}^{C}D_t^{\alpha} y(t) = F(y(t)), \quad t>0, \quad 0<\alpha<1 \tag{188}$$

Considering $F(y(t)) = ay - by^2$ we get a time-fractional logistic model with the Caputo derivative

$${}^{C}D_t^{\alpha} y(t) = ay - by^2, \quad t>0, \quad 0<\alpha<1 \tag{189}$$

The development of Eq.(188) (and particularity Eq. (189)) was possible only because the Caputo derivative is left inverse to the fractional integral. *If such a property does not hold with other types of operators, where the kernels are not of power law type, the hereditary model is* Eq.(185), which a general hereditary construction.

Note: We can see now, in contrast to the fractionalized model using the construction of the Volterra equation of the second kind, when the kernel $R(t;\alpha) \to 0$ for $t \to \infty$ there is no asymptotic state $y(t) \to 1$; we will see such behaviors in the solution of the surrogate fractional models discussed in Section 7.

Remark 9. *The time-fractional growth model with the Caputo derivative (Eq.(185)) is the only case when the correctly derived model and the formal fractionalized equation (by replacing $dy(t)/dt$ by ${}^{C}D_t^{\alpha} y(t)$) coincide. We can see many examples in Section 7 and the results thereof.*

7. Surrogate fractional models (obtained by replacements)

Now, we stress the attention on various approaches to solve logistic equations when the time fractional derivative in the Verhulst model is replaced by different fractional operators (derivatives); but we do not discuss their origins (derivations). A standpoint about the renaming all models considered in this section as *surrogate*, and some additional details, is expressed in Remark 16. There are two groups, in accordance with the operator used: Models with singular memories (Section 7.1) and models with non-singular memories (Sections 7.2, 7.3 and 7.4).

7.1. Fractional models with singular memory kernels

7.1.1. Fractional logistic equation with Caputo derivative: Analytic techniques

We start considering the fractional logistic equation with Caputo derivative since this is the dominating case, irrespective of the solution method applied [119–139, 145–147]. We will try to encompass the more important results and to compile an expository text that would be useful for understanding the main lines of solutions and emerging problems. That is, we have the general construction of the fractional logistic equation with Caputo derivative [145].

As it was commented in the preceding section dynamic models with the Caputo derivative belong to a class of models where the constructions obtained from basic principles (causality concept) and by formal replacements coincide. That is, we have the general construction of the fractional logistic equation with Caputo derivative [145].

$$D_t^{\alpha} y\left(t\right) = y\left(t\right)\left(1 - y\left(t\right)\right), \quad y\left(0\right) = y_0, \quad t > 0, \quad 0 < \alpha < 1 \tag{190}$$

Note: Since the focus now is on the solution techniques applied, we accept, for the sake of simplicity, that all coefficients in Eq.(190) are equal to unity.

- Direct integration approach [145]

Nieto [145] applied the fractional integral (Riemann-Liouville integral) to both sides of Eq.(190) to get

$$I_t^{\alpha}\left[D_t^{\alpha} y\left(t\right)\right] = I_t^{\alpha}\left[y\left(t\right)\left(1 - y\left(t\right)\right)\right] \tag{191}$$

and bearing in mind that $I_t^{\alpha}\left[D_t^{\alpha} y\left(t\right)\right] = y\left(t\right) + c$, where c is an arbitrary constant, the result of the integration is

$$y\left(t\right) - y_0 = \left(1 - \alpha\right)\left[y\left(t\right)\left(1 - y\left(t\right)\right) - y_0\left(1 - y_0\right)\right] + \alpha \int_0^t y\left(s\right)\left[1 - y\left(s\right)\right] ds \tag{192}$$

Then, differentiating both sides of Eq.(193) we obtain

$$y\left(t\right) - y_0 = \left(1 - \alpha\right)\left[y\left(t\right)\left(1 - y\left(t\right) - y\left(t\right)\frac{dy\left(t\right)}{d}\right)\right] + \alpha y\left(t\right)\left[1 - y\left(t\right)\right] \tag{193}$$

or more compactly as

$$\alpha\frac{dy\left(t\right)}{dt} + 2\left(1 - \alpha\right) y\frac{dy\left(t\right)}{dt} = \alpha y\left(t\right)\left[1 - y\left(t\right)\right] \tag{194}$$

From Eq.(194) we recover the ordinary Verhulst equation for $\alpha = 1$.

Equation (194) can be rearranged as (for $y \neq 0, 1$)

$$\alpha\frac{dy}{dt} - 2\alpha y\frac{dy}{dt} + 2y\frac{dy}{dt} = \alpha y\left(1 - y\right) \Rightarrow \frac{dy/dt - 2y dy/dt}{y - y^2} + \frac{2}{\alpha}\frac{dy/dt}{1 - y} \tag{195}$$

or

$$\frac{d}{dt}\ln\left|1 - y^2\right| - \frac{2}{\alpha}\ln\left|1 - y\right| = 1 \Rightarrow \ln\left|1 - y^2\right| - \ln\left(\left|1 - y\right|^{2/\alpha}\right) = t + c \tag{196}$$

Finally we get

$$\frac{y - y^2}{\left(1 - y\right)^{2/\alpha}} = \frac{y - y_0^2}{\left(1 - y_0\right)^{2/\alpha}} \exp\left(t\right) \tag{197}$$

- Power series solution [140]

Area and Nieto [140] developed a power series solution by the expansion, so that

$$y\left(t\right) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow \frac{dy\left(t\right)}{t} = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty}\left(n + 1\right) a_{n+1} t^n \tag{198}$$

Then, from the integer order model (the Verhulst equation), the following recurrence relation appears

$$a_{n+1} = \frac{1}{n + 1}\left[a_n - \sum_{j=0}^{n} a_j a_{n-j}\right], n \geq 1, a_0 = y_0 = \frac{1}{2} \tag{199}$$

That is, with $a_0 = 1/2$, $a_1 = 1/4$, $a_2 = 0$, $a_3 = -1/48$,$a_4 = 0$, $a_3 = 1/480$ ... we obtain the approximate solution of the Verhulst equation [140]

$$\frac{1}{1+e^{-t}} \approx \frac{1}{2} + \frac{1}{4}t - \frac{1}{48}t^2 + \frac{1}{480}t^3 \tag{200}$$

Remark 10 (On the recurrence relation (199))**.** *Area and Nieto [141] noted that the same relation can be obtained in a slightly different way, precisely as briefly explained next. It needs to be noted that the initial coefficient, $a_0 = 1/2$, is chosen from the local form of the logistic equation.*
With the assumption that the solution can be developed as a series (198), then with derivative $dy(t)/dt$ (from the same expression (198)) and with (a multiplication of infinite series (recall the remark 7 explaining the appearance of convolution on the Zwanzig solution)

$$y^2(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} (n+1) a_j a_{n-j} \right) t^n \tag{201}$$

Then, substituting $y(t)$ and $dy(t)/dt$ in the classical Verhulst model, the recurrence (199) can be obtained. Alternatively, the same recurrence relation can be obtained by applying the Laplace transform [141] to the right-hand side of the logistic model, so that

$$\mathcal{L}\left[y(t) - y^2(t)\right] = \sum_{n=0}^{\infty} (a_n - b_n) \frac{n!}{s^{n+1}}, \quad b_n = \sum_{j=0}^{n} a_j a_{n-j} \tag{202}$$

as well as with

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) = \sum_{n=0}^{\infty} a_{n+1}(n+1)\frac{n!}{s^{n+1}} \tag{203}$$

The result in the Laplace domain is

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)\frac{n!}{s^{n+1}} = \sum_{n=0}^{\infty} (a_n - b_n) \frac{n!}{s^{n+1}} \tag{204}$$

That directly leads to the recurrence (199); to start the recurrence, it is needed $a_0 = y(0)$ (see Eq.(199)).

- Power series solution in terms of the Euler polynomials [140]

If the Euler polynomials are considered (see Section 2.1.2 about the Taylor series expansion), i.e.,

$$\frac{2\exp(\xi t)}{\exp(t) + 1} = \sum_{n=0}^{\infty} E_n(\xi) \frac{t^n}{n!} \tag{205}$$

and the series is convergent for $|t| < \pi$.
Thus, for $\xi = 1$,

$$\frac{2\exp(t)}{\exp(t)+1} = \frac{2}{1+\exp(-t)} = \sum_{n=0}^{\infty} E_n(1)\frac{t^n}{n!} \tag{206}$$

Hence, we have the approximation

$$\frac{1}{1+\exp(-t)} = \frac{1}{2}\sum_{n=0}^{\infty} E_n(1)\frac{t^n}{n!} \tag{207}$$

Therefore, the coefficients in the power series expansion can be represented by the Euler numbers as $a_n = \frac{1}{2}\frac{E_n(1)}{n!}$. Since, for the logistic function $a_n = y^{(n)}(0)/n!$, we have the relation $a_n = E_n(1)/2 = y^{(n)}(0)$ [140].
After the initial steps demonstrating the main logic of the power series solution, we look at the fractional logistic equation (190) and try to construct the solution as $x(t) = \sum_{n=0}^{t} b_n(\alpha)(t^{\alpha})^n$. Skipping details, we get a recurrence relation [140]

$$b_{n+1}(\alpha) = \frac{\Gamma(n\alpha+1)}{\Gamma[(n+1)\alpha+1]}\left[b_n(\alpha) - \sum_{j=0}^{n} b_j(\alpha) b_{n-j}(\alpha)\right], \quad n > 0 \tag{208}$$

Hence, for example: $b_0(\alpha) = 1/2$, $b_2(\alpha) = 0$, $b_3(\alpha) = \Gamma(2\alpha+1)/\left[16\Gamma(\alpha+1)^2\right]\Gamma(3\alpha+1)$, $b_4(\alpha) = 0$.

- Logistic model with Caputo derivative and Allee effects

Now, consider a logistic model with Allee effect [142]

$$\frac{dy(t)}{dt} = y(t)(1-y(t))[y(t)-A], \quad A > 0 \tag{209}$$

with constant solutions $0, 1, A$.

For $\frac{dy}{dt} > 0$ *and when* $A < y(0) < 1$ *we have an increasing* $y(t)$

For $\frac{dy}{dt} < 0$ *and when* $0 < y(0) < A$, *then* $y(t)$ is decreasing

With the assumption of developing a solution as a series (198) we have $y^2(t)$ approximated by Eq.(201) and $y^3(t)$ as

$$y^3(t) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{k=0}^{j} a_k a_{n-j}\right)\right)t^n \tag{210}$$

The substitution of these formal series approximations into Eq.(209) and bearing in mind that in such a case

$${}^{C}D_t^{\alpha} \approx \sum_{n=0}^{\infty} b_{n+1}(\alpha)\frac{\Gamma[(n+1)\alpha+1]}{\Gamma(n\alpha+1)}(t^{\alpha})^n \tag{211}$$

The following recurrence relation about the coefficients $b_n(\alpha)$ appears

$$b_{n+1} = \frac{\Gamma(n\alpha+1}{\Gamma[(n+1)\alpha+1]}\left[-Ab_n + (A+1)\sum_{j=0}^{n} b_j b_{n-j} - \sum_{j=0}^{n}\left(\sum_{k=0}^{j} b_k b_{j-k}\right)b_{n-j}\right] \tag{212}$$

valid for $n > 0$ and an initial condition $b_0 = y(0)$.

Note: A detailed analysis of stability and bifurcation behavior of a fractional logistic model with Allee effect was done in [143] oriented to thresholds to extinction of low-density populations (see also the references therein).

Remark 11 (On the exact solution of the fractional logistic equation)**.** *Area et al. [128] considered an analysis provoked by the preceding solution of West [127], who declared a development of an exact solution of Eq.(190) applying the Carleman embedding technique (see details about the solutions of a quasi Verhulst equation in [148]- Chapter 3.2) , as*

$$y(t) = \sum_{n=0}^{\infty}\left(\frac{y_0-1}{y_0}\right)^n E_{\alpha}(-nt^{\alpha}) \tag{213}$$

The detailed analysis of Area et. al. [128] demonstrated that the solution of West is valid only when $\alpha = 1$, *but suggested that Carleman techniques would be promising for solutions of fractional equations, although still undeveloped in this direction. D'Ovidio et al. [134] tested the function (213) but with a modified fractional logistic equation*

$$D_t^{\alpha} y(t) = y(t)(1-y(t)) + y_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \int_0^{\infty}\int_0^{\infty}\left[y(s)\,y(z) - y^2(s)\right] l_{\alpha}(z,t)\,dsdz \tag{214}$$

The additive term is related to the solution of the classical logistic equation. Also, the function obeys the conditions $l_{\alpha}(0,t) = t^{-\alpha}/\Gamma(1-\alpha)$ *and* $\int_0^{\infty} l_{\alpha}(s,t)ds = 1$, *as well as its Laplace transform, is* $\int_0^{\infty} e^{-\lambda s} l_{\alpha}(s,t)ds = E_{\alpha}(-\lambda t^{\alpha})$, $\lambda > 0$. *The tests in [134] demonstrated that (213) is a solution of Eq. (214), but not of Eq. (190).*

Ortigueira and Bengochea [130] were also attracted by the discussion on the solution of the fractional logistic equation raised by Area et al. [128] and solved the problem from a different point of view: they suggested that the fractional derivative is the Grunwald-Letnikov (GL) derivative, since, in contrast to the

Caputo derivative, there is no restriction the function to be differentiable. Assuming a solution as a Taylor series and the Pade approximation technique, a solution was developed.

Currently, since there are no exact solutions to the fractional logistic equation, many attempts are being made to develop approximate solutions using various techniques such as spectral methods [124, 132], as the series approximations discussed above. As an exception, Tarasov [137] developed exact solutions to Bernoulli and fractional logistic models in special cases where the coefficient is of power law, with examples from mechanics and economics.

Remark 12 (The Euler number approach to fractional logistic equation solution)**.** *Efficient solutions to the fractional logistic equations have been developed in [133, 138] by applying the Euler number approach, like that demonstrated earlier. A more advanced study in this direction was developed by Nieto [146].*

Remark 13 (Numerical techniques)**.** *Various numerical techniques have been applied to solve the fractional logistic equation Eq.(190), among them: Optimal HAM [125], iterative methods [119, 121], collocation methods [124, 129, 131], Chebishev approximations [123] , Adams-type predictor-corrector method [122], spectral methods [132, 136], Padé approximants [120], etc.; see for more details the quoted works and the reference therein. We do not develop analyzes of this trend in solutions of fractional logistic equations since it is beyond the framework of the study, but this does not mean you should neglect it.*

Note: For completeness of the reference information we refer to the thorough analysis and strong results concerning approximate analytical methods with Caputo fractional derivative developed by Aibinu and Momoniat [155].

7.1.2. Λ-fractional logistic equation

- Formulation and a simple series solution [150]

Recently, Jornet and Nieto [150] considered a fractional-logistic equation based on the Λ-fractional derivative [151–153]

$$^{\Lambda}D^{\alpha}y = y\,(1-y) = y - y^2, \quad t>0, \quad 0<\alpha<1 \tag{215}$$

where

$$^{\Lambda}D^{\alpha}y = \frac{^{C}D^{\alpha}y\,(t)}{^{C}D^{\alpha}t} \tag{216}$$

is the Leibniz fractional derivative of an absolutely continuous function $y \in [0,T] \to \mathbb{R}$ [151]. In the nominator and denominator of $^{\Lambda}D^{\alpha}y$ there are Caputo fractional derivatives.

Equation (215) is equivalent to a non-autonomous Caputo equation [150]

$$^{C}D_t^{\alpha}y = \frac{t^{1-\alpha}}{\Gamma\,(2-\alpha)}y\,(1-y) \tag{217}$$

The $^{\Lambda}D^{\alpha}y$ derivative defines a differential, and therefore, it has a geometrical meaning [150–152].

Assuming a solution as a formal series expansion $y\,(t) = \sum\limits_{0}^{\infty} a_n t^n$, a recursive relation was obtained

$$a_{n+1} = \frac{\Gamma\,(n+2-\alpha)}{\Gamma\,(n+2)\,\Gamma\,(2-\alpha)}\left[a_n - \sum_{m=0}^{n} a_m a_{n-m}\right] \tag{218}$$

Besides, the solution in terms of $^{\Lambda}D^{\alpha}y$ is C^{∞} and real analytic, expanded in terms of t^n instead $t^{\alpha n}$, in contrast to the case when the logistic equation uses the Caputo derivative [150]. Further, as demonstrated by Jornet and Nieto [150]

$$^{\Lambda}D^{\alpha}y = \frac{dy\,(0)}{dt} + \frac{1}{t^{1-\alpha}}\int\limits_0^t \frac{1}{(t-s)^{\alpha-1}}\frac{d^2y\,(s)}{ds^2}ds \tag{219}$$

showing that it is with *a singular kernel*.

Since with $y = C$ (a constant) we have $^{\Lambda}D^{\alpha}C = 0$, the initial condition for Eq.(215) is the same as in the case when the Caputo derivative is applied [150] .

Last, but not least, considering the concept to represent the solution to Eq.(215) *via* the Euler numbers, the coefficients a_n of the series solution (defined by Eq.(218)) relate as

$$E_n^{\alpha} = 2a_n^{\alpha}\Gamma(2-\alpha)^n \prod_{j=1}^{n} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \tag{220}$$

When ${}^{\Lambda}D^{\alpha}y(t)$ is applied to the population equation, there is a possibility that the local (integer-order) model to be extended to a non-local (fractional) version [150]. More interesting details about the properties of ${}^{\Lambda}D^{\alpha}y(t)$ are available in [150], especially the singularity of its kernel.

- More details on the Λ-fractional derivative and the logistic equation

The Λ-fractional derivative was defined above, but now, for the sake of clarity of exposition, we would like to present some additional details. As mentioned, in fact, it is the Leibniz fractional derivative defined by Eq.(216) as ${}^{\Lambda}D_t^{\alpha}y(t) = \frac{{}^{C}D_t^{\alpha}y(t)}{{}^{C}D_t^{\alpha}t}$ (see below)

With the Riemann-Liouville derivative of order $\alpha \in (0,1)$

$$D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{u(s)}{(t-s)^{\alpha}}ds, \quad \alpha \in (0,1), \quad t \in [0,T] \tag{221}$$

and, as well as with $n \in N$, we have

$$D_t^{\alpha}t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha} \Rightarrow D_t^{\alpha}t^{\gamma}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{n-\alpha}, \quad \alpha > 0, \quad \gamma > -1 \tag{222}$$

For $\gamma = 2$ we have $D^{\gamma}t^2 = \frac{\Gamma(3)}{\Gamma(3-\gamma)}t^{2-\gamma} = \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)}$.

Now, if we need to differentiate a power-law function, i.e., to find ${}^{\Lambda}D_t^{\alpha}t^n$, applying the definition we have [154]

$${}^{\Lambda}D_t^{\alpha}t^n = \frac{\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}}{\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}} = \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}t^{n-1} \tag{223}$$

It is obvious that for $\alpha \to 1$ we have ${}^{\Lambda}D_t^{\alpha}t^n{}_{(\alpha\to 1)} \Rightarrow \frac{dt^n}{dt} = nt^{n-1}$. To make it clear, let us explain the transformation from the linear t space into the Λ-space [154], that is, this transformation is

$$t \to T = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)} \tag{224}$$

Precisely, let us focus on the space transformation $u(t) \to U(T)$ as well as the power-law transform $t^b \to T^B$, which enables us to find a simple form of the logistic equation (see further Eq.(227). From the definition of T we have that $t = T\left(\Gamma(3-\gamma)^{\frac{1}{2-\gamma}}\right)$. Moreover,

$$\begin{gathered} T^B = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{s^b}{(t-s)^{\gamma}}ds = \frac{\Gamma(1+b)\,t^{1+b-\gamma}}{\Gamma(2+b-\gamma)} \Rightarrow \\ \Rightarrow T^B = \frac{\Gamma(1-b)\left[\Gamma(3-\gamma)T\right]^{\frac{1+b-\gamma}{2-\gamma}}}{\Gamma(2+b-\gamma)} \end{gathered} \tag{225}$$

With these preliminaries, we may consider the Λ-logistic equation presented as (in the original notations of [154]) (for $T > 0$, $u \in AC[0,T]$)

$${}^{\Lambda}D_t^{\alpha}u(t) = t^b u(1-u) \tag{226}$$

where $0 < \gamma < 1$ in the initial t-space and take into account that the coefficient is t^b, i.e., time-dependent. Therefore, in the Λ-space, we have

$$\frac{dU(T)}{dT} = AT^B\left[U(T) - U^2(T)\right], \quad B = \frac{1+b-\gamma}{2-\gamma} \tag{227}$$

where $A = \frac{\Gamma(1-b)[\Gamma(3-\gamma)]^{\frac{1+b-\gamma}{2-\gamma}}}{\Gamma(2+b-\gamma)}$.

Then, the solution obtained (by Wolfram Mathematica) [154] in the Λ-space is

$$U(T) = \frac{1}{1 + C \exp\left[-\left(\frac{A}{B+1}\right) T^{B+1}\right]} \tag{228}$$

where C is a constant.

Now, going back to the initial t space we have $\bar{U}(t) = U\left[T \to \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right]$ and the solution is

$$u(t) = D^{1-\gamma}\bar{U}(t) = \frac{1}{\Gamma(\gamma)} \frac{d}{dt} \int_0^t \frac{\bar{U}(s)}{(t-s)^{1-\gamma}} ds =$$

$$= \frac{\exp\left[\frac{A\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)}{B+1}\right] \left\{\gamma t^{\gamma}\Gamma(3-\gamma)\left[\exp\left[\frac{A\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)}{B+1}\right] + C\right] - AC(\gamma-2)\, t^2 \left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)^B\right\}^B}{\gamma t \Gamma(3-\gamma)\Gamma(\gamma)\left[\exp\left[\frac{A\left(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\right)}{B+1}\right] + C\right]^2} \tag{229}$$

Remark 14 (Some comments on the Λ-fractional logistic equation)**.** *We can see that the transforms imposed by $D_t^{\alpha} y(t)$ allow us to obtain a logistic equation (227) very close to the original Verhulst model, which is easy to solve. However, the inverse transform into the original t space is not easy. Moreover, as also commented in [154], the solution of Eq.(229) it is to complicated and hard to interpret, compared with, for example, the series solutions, as was demonstrated above by the recursive relation (218). The solutions developed in [150] and [154], and discussed here, are examples of what we can do with a newly defined derivative. However, we have to expect more easy-to-understand applications of $D_t^{\alpha} y(t)$, since there is a serious background for this in [151–153].*

7.2. Fractional models with non-singular memory kernels

Now, we turn on non-local models with fractional operators with non-singular memories, a trend that began in 2015 when the Caputo-Fabrizio derivative was conceived [156]. This was the nuclei expanded towards the definition of the Atangana-Baleanu derivative (ABC) based on the Mittag-Leffler kernel [157] in 2016, and attempts to generalize these operators as special cases of the Prabhakar fractional calculus [158]. The following parts of this section consider models starting with a simple case with the Caputo-Fabrizio derivative and gradually passing through models with ABC and Prabhakar derivatives.

7.2.1. Logistic equations with the Caputo-Fabrizio operator

- Solution of Kumar et al. [159]

Kumar et al. [159] considered a fractional logistic equation with the Caputo-Fabrizio derivative (231) with a non-singular kernel [156]

$$^{CF}D_t^{\alpha} y(t) = ky(t)(1 - y(t)), \quad t > 0, \quad y(0) = y_0 = a \neq 0 \tag{230}$$

$$^{CF}D_t^{\alpha} y(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\varepsilon)} \frac{dy(s)}{dt} ds \tag{231}$$

Stability issues:

The stability at the equilibrium points $y = 0, 1$, with

$$\frac{dg}{dt}(y(t)) = k(1 - 2y(t)) \Rightarrow \frac{dg}{dt}(0) = k, \quad \frac{dg}{dt}(1) = -k \tag{232}$$

Then, the solution of

$$^{CF}D_t^{\alpha} \varepsilon(t) = g'\left(y_{eq} = 0\right) \varepsilon(t) = k\varepsilon(t), \quad t > 0, \quad \varepsilon(0) = y_0 \tag{233}$$

is

$$\varepsilon(t) = \frac{y_0}{1 - k + k\alpha} \exp\left(\frac{k\alpha}{1 - k + k\alpha}\right) \tag{234}$$

demonstrating that the point $y = 0$ is unstable. Otherwise, at $y = 1$, the solution of

$$^{CF}D_t^{\alpha}\varepsilon(t) = g'\left(y_{eq} = 1\right)\varepsilon(t) = -k\varepsilon(t), t > 0, \varepsilon(0) = y_0 - 1 \tag{235}$$

is

$$\varepsilon(t) = \frac{y_0 - 1}{1 - k + k\alpha}\exp\left(\frac{k\alpha}{1 - k + k\alpha}\right) \tag{236}$$

and reveals that the equilibrium point $y = 1$ is asymptotically stable.

Uniqueness of the solution:

The analysis in [159] reveals that if the condition (237) is satisfied (with $M(\alpha) = 1$)

$$\left(1 - k\frac{2(1-\alpha)}{(2-\alpha)} - kt\frac{2\alpha}{(2-\alpha)}\right) > 0 \tag{237}$$

then, Eq.(230) has a unique solution.

A numerical solution based on the perturbation techniques and Pade approximation, with $y_0 = 1/2$ (two values of the coefficient k ($k = 1/2$ $k = 1/3$) was used to calculate plots showing the effect of the fractional order α.

- A hybrid solution of Jafari et al. [160]

Jafari et al. [160] also considered Eq.(230) and its solution, at least the initial steps applied, to some extent, resemble the approach used by Nieto (see above). Precisely, first rearranging Eq.(230) as

$$\begin{aligned} &\frac{M(\alpha)}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}\frac{dy(s)}{ds}ds = f(t, y(t)) \Rightarrow \\ &\Rightarrow \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}\frac{dy(s)}{ds}ds = \frac{1-\alpha}{M(\alpha)}e^{-\frac{\alpha}{1-\alpha}t}f(t, y(t)) \end{aligned} \tag{238}$$

Then, differentiation applied to both sides of Eq.(238) yields

$$\frac{dy(t)}{dy} = \frac{1-\alpha}{M(\alpha)}\left[\frac{df(t, y(t))}{dt} + \frac{\alpha}{1-\alpha}f(t, y(t))\right] \tag{239}$$

With $f(0, y(0)) = 0$, the solution of Eq.(239) is

$$y(t) = y(0) + \frac{1-\alpha}{M(\alpha)}f(t, y(t)) + \frac{\alpha}{M(\alpha)}\int_0^t f(s, y(s))ds \tag{240}$$

Further, expressing Eq.(240) for two different times $t = t_m$ and $t = t_{m+1}$, and subtracting them, side by side, the result (the difference) is

$$y(t_{m+1}) - y(t_m) = \frac{1-\alpha}{M(\alpha)}\left(f(t_m, y(t_m)) - f(t_{m+1}, y(t_{m+1}))\right) + \frac{\alpha}{M(\alpha)}\int_{t_m}^{t_{m+1}} f(s, y(s))ds \tag{241}$$

Applying the three-step Adams-Bashforth technique to the integral term and using a Lagrange interpolation of $f(s, y(s))$on $[t_m, t_{m+1}]$, the solution was developed.

- A series solution of Area and Nieto [141]

Consider Eq.(230) with the series solution approximation (198), and applying the Laplace transform to both sides we get

$$\sum_{n=1}^{\infty}\frac{\Gamma(n\xi+1)}{(\alpha(1-s)+s)s^{n\xi}} = \sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(n\xi+1)}{s^{n\xi+1}} \tag{242}$$

For example, if $\xi = \alpha$ and when $\alpha \to 1$ (from the left),), then we get the recursive relation (199).

The relationship (242) can be rearranged as (see the steps in [141])

$$\sum_{n=1}^{\infty}\Gamma(n\xi+1)\frac{a_n}{s^{n\xi}}=\alpha\sum_{n=0}^{\infty}(a_n-b_n)\frac{\Gamma(n\xi+1)}{s^{n\xi+1}}+ \\ +(1-\alpha)\sum_{n=0}^{\infty}(a_n-b_n)\frac{\Gamma(n\xi+1)}{s^{n\xi}} \tag{243}$$

When $\xi=1$, then Eq.(243) reduces to

$$a_n=\frac{1}{n}\left(a_{n-1}-b_{n-1}+nb_n\frac{\alpha-1}{\alpha}\right) \tag{244}$$

and for $\alpha\to1$ we get the recurrence (199).

- Laplace transform solution by Kanth and Garg [144]

Consider again Eq.(230) in a simplified form [144]

$$^{Cf}D_t^{\alpha}x(t)=k(1-x(t)),\quad 0<\alpha<1 \tag{245}$$

and applying the Laplace transform to both sides, we get

$$X(s)=\frac{x(0)-1}{(1+k-\alpha k)(s+\frac{\alpha k}{1+k-\alpha k})}+\frac{1}{s} \tag{246}$$

taking into account that $\mathcal{L}[^{Cf}D_t^{\alpha}x(t)]$ (see the definition (231)) is [156]

$$\mathcal{L}[^{Cf}D_t^{\alpha+n}x(t)]=\frac{s^{n+1}\mathcal{L}[x(t)]-x(0)S^n-x(0)s^{n-1}-\ldots x^{(n)}(0)}{s+(1-s)\alpha} \tag{247}$$

Then, the inverse transform of Eq.(246) yields

$$x(t)=\frac{x(0)-1}{1+k(1-\alpha)}exp\left[-\frac{\alpha k}{1+k(1-\alpha)}t\right]+1 \tag{248}$$

Now, with the substitutions $y(t)=1/x(t)$ and $x(0)=1/y(0)$ we obtain

$$y(t)=\left\{1+\frac{1-y(0)}{y(0)}\frac{1}{1+k(1-\alpha)}exp\left[-\frac{\alpha k}{1+k(1-\alpha)}t\right]\right\}^{-1} \tag{249}$$

For $\alpha\to1$, Eq.(249) approaches to the integer-order Verhulst model, namely

$$y(t)_{\alpha\to1}=\left[1+\left(\frac{1-y(0)}{y(0)}\right)exp(-kt)\right]^{-1} \tag{250}$$

Note: For completeness of the exposition, even though all published results cannot be encompassed here, we have to mention the work of Djeddi et al. [161] where a modified analytical approach to generalized logistic models with quadratic and cubic terms was developed. Attempts to solve the logistic models with the Caputo-Fabrizio derivative, by numerical solutions, applying a successive approximation method were reported by Adel and Khader [162]. Also, and use the compact finite difference scheme to solve this equation numerically was developed by Sattari and Ameri [163]; we mention these studies for completeness of the reference quotations since they also belong to the group of fractional models obtained by replacement.

7.3. Logistic dynamics in the Mittag-Leffler memory fractional framework

7.3.1. Logistic model with the Atangana-Baleanu derivative

- A solution by application of the Atangana-Baleanu integral [164]

Hassan et al. [164] considered the Verhulst logistic model

$$^{ABC}D_{\alpha}^{t}=ky\left(t\right)\left(1-y\left(t\right)\right),\quad 0<\alpha\leq1,\quad t\geq0,\quad k>0,\quad y\left(0\right)=y_0>0 \tag{251}$$

by applying the Atangana-Baleanu fractional derivative of Caputo type [157]

$$ {}^{ABC}D_{\alpha}^{t}f(t)=\frac{M(\alpha)}{1-\alpha}\int_{0}^{t}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right)\frac{df(s)}{ds}ds= $$
$$ =\frac{M(\alpha)}{1-\alpha}\frac{df(t)}{dt}*E_{\alpha}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right), \tag{252} $$
$$ E_{\alpha}(-t^{\alpha})=\sum_{0}^{\infty}\frac{(-1)^{n}t^{\alpha n}}{\Gamma(\alpha n+1)} $$

where $E_{\alpha}(-t^{\alpha})$ is the Mittag-Leffler function with one parameter, and the condition imposed on $M(\alpha)$ is $M(0)=M(1)=1$.

The Laplace transforms of ${}^{ABC}D_{\alpha}^{t}f(t)$ is

$$ \mathcal{L}\left[{}^{ABC}D_{t}^{\alpha}f(t)\right]=\frac{M(\alpha)}{1-\alpha}\frac{\left(s^{\alpha}F(s)-s^{\alpha-1}f(0)\right)}{s^{\alpha}+\frac{\alpha}{1-\alpha}} \tag{253} $$

Particularly, with a power-law function $f(t)=t^{n}$, pertinent to the series solution developed , is

$$ \mathcal{L}\left[{}^{ABC}D_{t}^{\alpha}t^{n}\right]=\frac{M(\alpha)}{(1-\alpha)}\frac{1}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{\Gamma(n+1)}{s^{n-\alpha+1}} \tag{254} $$

The model (251) was investigated in the Sobolev space. Moreover, a detailed analysis based on the fixed-point theory was done towards existence and uniqueness (see details in the original work).

As a first step of the solution technique, the Atangana-Baleanu Integral ${}^{ABC}I_{t}^{\alpha}$ (see Remark 15 for explanations)

$$ {}^{ABC}I_{t}^{\alpha}f(t)=\frac{1-\alpha}{M(\alpha)}f(t)+\frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{a}^{t}\frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t>a \tag{255} $$

was applied to both sides of Eq.(251) to obtain

$$ y(t)=y_{0}+\frac{1-\alpha}{M(\alpha)}ky(t)(1-y(t))+\frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{0}^{t}ky(s)(1-y(s))\frac{1}{(t-s)^{1-\alpha}}ds \tag{256} $$

Denoting $Y(t)=y(t)(1-y(t))$, a more compact form of Eq.(255) for further analysis was used

$$ y(t)=y_{0}+\frac{1-\alpha}{M(\alpha)}[Y(t)]+\frac{\alpha}{M(\alpha)}\int_{0}^{t}Y(s)\frac{1}{(t-s)^{1-\alpha}}ds \tag{257} $$

Equation (257) was proved for the existence of unique solutions with $y_{0}\neq 0$ following the Banach fixed point theorem (more details in the original work).

Remark 15 (On the Atangana-Baleanu integral)**.** *As was demonstrated by Atangana and Baleanu [157] the following fractional differential equation*

$$ {}_{0}^{ABR}D_{t}^{\alpha}[f(t)]=u(t) \tag{258} $$

where ${}_{0}^{ABR}D_{t}^{\alpha}f(t)$ *is the Atangana-Baleanu derivative in the Riemann-Liouville sense.*

$$ {}^{ABR}D_{a+}^{\alpha}f(t)=\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_{0}^{z}f(z)E_{\alpha}\left[\frac{-\alpha}{1-\alpha}(t-z)^{\alpha}\right]dz, \quad 0<\alpha<1, $$
$$ a<t<b, \quad f(t)\in L^{1}[a,b] \tag{259} $$

and

$$ {}^{ABC}D_{t}^{\alpha}[f(t)]={}_{0}^{ABR}D_{t}^{\alpha}[f(t)]-\frac{M(\alpha)}{1-\alpha}f(0)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right) \tag{260} $$

Equation (258 has a unique solution

$$f(t) = \frac{1-\alpha}{M(\alpha)} u(t) + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t u(\tau)(t-\tau)^{\alpha-1} d\tau \tag{261}$$

and from this, the associated Atangana-Baleanu integral was defined (with $M(\alpha) = 1$, for the sake of simplicity of exposition)

$$^{AB}I_{a+}^{\alpha} u(t) = (1-\alpha) u(x,t) + \alpha\, ^{RL}I_{a+}^{\alpha} u(x,t) = f(x,t) \tag{262}$$

where for $\alpha = 0$ we recover the initial function, while for $\alpha = 1$ we get the ordinary Riemann integral.

Further, we have

$$^{AB}I_{a+}^{\alpha} \left[^{ABR}D_{a+}^{\alpha} f(t) \right] = f(t), \quad ^{ABR}D_{a+}^{\alpha} f(t) \left[^{AB}I_{a+}^{\alpha} f(t) \right] = f(t) \tag{263}$$

***Note:** The derivative used in Eq.(252) and the operation resulting in Eq.(256) is of Caputo type, i.e., ${}_0^{ABC}D_t^{\alpha}$. A detailed analysis of the associated and constitutive Atangana-Baleanu integrals (as well as of the Caputo-Fabrizio operators) is available in [105].*

Two numerical approaches were applied to solve Eq.(251): Successive substitution algorithm (SSA) and the reproducing (RK) method, briefly presented next.

1) *Solution by the successive substitution algorithm (SSA):*

Representing Eq.(256) as an equivalent Volterra integral equation, the following recurrence formula was developed [164]

$$\begin{aligned} y_{n+1}(t) &= y_0 + \frac{1-\alpha}{M(\alpha)} k \left[y_n(t) \left(1 - y_n(t)\right)\right] + \\ &+ \frac{\alpha}{M(\alpha)} k \int_0^t y_n(s) \left(1 - y_n(s)\right) \frac{1}{(t-s)^{1-\alpha}} ds \end{aligned}, \quad n = 0, 1, 2 \ldots \tag{264}$$

with $y_{n=0} = y_0$

Note: It is easy for the reader to see the logical line to use the heredity of the logistic model expressed as Volterra integral equations, as we commented earlier.

2) *Reproducing kernel (RK) method:* As a first step, the initial condition was homogenized as $Z(t) = y(t) - y_0$. This allows us to get

$$^{ABC}D_t^{\alpha} \left[Z(t) + y_0 \right] = k \left(Z(t) + y_0 \right) \left(1 - Z(t) - y_0 \right) \tag{265}$$

taking into account that $^{ABC}D_t^{\alpha} y_0 = 0$.

Then, the solution of the following initial value problem was developed

$$^{ABC}D_t^{\alpha} Z(t) = k \left(Z(t) + y_0 \right) \left(1 - Z(t) - y_0 \right), \quad Z(0) = 0, \quad t > 0 \tag{266}$$

Definition of the differential operator (in the original notations) ($L : W_2^2[a,b] \rightarrow W_2^1[a,b]$ (see the note below), such that $L[Z(t)] = {}^{ABC}D_t^{\alpha} Z(t)$, allows to rewrite Eq.(266) as

$$^{ABC}D_t^{\alpha} Z(t) = k \left[Z(t) + y_0 \right] \left[1 - Z(t) - y_0 \right], \quad t > 0 \tag{267}$$

Note: The operator L is bounded and linear from $W_2^2[a,b]$to $W_2^1[a,b]$, so that $L[Z(t)] = {}^{ABC}D_t^{\alpha} Z(t)$ (the proof is provided by Theorem 4 in [164])

3) *Numerical simulations:*

Numerical solutions were carried out with different values of y_0, precisely $y_0 = 1/2$, $y_0 = 1/4$ and $y_0 = 0$,compared with exact solutions $y(t) = \frac{1}{1+e^{-y_0 t}}$.

More details about the numerical procedures are available in the original work, but the results reveal that the plotted solutions are growing in time, and to some extent, resemble the original logistic function when $\alpha \rightarrow 1$, but no further analyzes are available, since the focus was on solution techniques, not on the physical relevance of the obtained results.

7.3.2. Solutions by the Laplace transform

As step towards a solution of Eq.(251), Bas and Ozarslan [165] considered the simple first-order kinetic model

$$^{ABC}D_t^{\alpha} = -ky(t), \quad t \geq 0, \quad 0 < \alpha < 1 \tag{268}$$

Applying the Laplace transform to both sides of Eq.(268), and rearranging the expressions to allow the inverse transform, the result is

$$y(t) = \frac{M(\alpha)}{1-\alpha} \frac{y(a)}{M(\alpha) - k(1-\alpha)} E_{\alpha} \left(\frac{k\alpha}{M(\alpha) - k(1-\alpha)} \right) \tag{269}$$

Note: In Eq.(269), the lower terminal in the convolution integral of (252) is defined as a .

The result (269) can be simplified, if $a = 0$ and $M(\alpha) = 1$, for the sake of simplicity, that is

$$y(t) = \frac{1}{1-\alpha} \frac{y(a)}{1 - k(1-\alpha)} E_{\alpha} \left(\frac{k\alpha}{1 - k(1-\alpha)} \right) \tag{270}$$

Then, the same approach was applied to Eq.(251) and the result (skipping the cumbersome expressions and transformations) is

$$\begin{gathered} y(t) = -k \frac{1-\alpha}{1-k(1-\alpha)} E_{\alpha} \left(\frac{k\alpha}{1-k(1-\alpha)} \right) t^{\alpha} + \\ + \left[\frac{1-\alpha}{1-k(1-\alpha)} \right]^2 \left(1 - E_{\alpha} \frac{k\alpha}{1-k(1-\alpha)} t^{\alpha} \right) + \frac{y(0)}{1-k(1-\alpha)} E_{\alpha} \left(\frac{k\alpha}{1-k(1-\alpha)} t^{\alpha} \right) \end{gathered} \tag{271}$$

in a simplified form, with $M(\alpha) = 1$ and $y(a) = y(0)$, allowing a more unified presentation, as it was done in many cases considered in this work.

7.3.3. A solution by a series

Consider Eq.(251) and a formal series solution $y(t) = \sum_0^{\infty} a_n t^n$, and then applying the Laplace transform we have [154] (see similar operations with the power series approximation in the preceding parts of this text)

$$\mathcal{L}\left[{}^{ABC}D_t^{\alpha} y(t) \right](s) = \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(n\xi + 1)}{\xi^{n\alpha+1}} \tag{272}$$

Then, for $\xi = \alpha$

$$\frac{M(\alpha)}{(1-\alpha)\left(s^{\alpha} + \frac{\alpha}{1-\alpha}\right)} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n\alpha+1)}{s^{(n-1)\alpha} + 1} = \sum_{n=0}^{\infty} (a_n - b_n) \frac{\Gamma(n\alpha+1)}{s^{n\alpha} + 1} \tag{273}$$

Further, with $b_n = a_0 a_n + \sum_{j=1}^{n-1} a_j a_{n-j}$, the recurrence relation is (we skip the intermediate steps that can be seen in details in [154])

$$a_n = \frac{(1-\alpha) \sum_{j=1}^{n-1} a_j a_{n-j} + \frac{\alpha \Gamma[\alpha(n-1)+1] \left[\sum_{j=1}^{n-1} a_j a_{n-j} + (2a_0 - 1) a_{n-1} \right]}{\Gamma(\alpha n + 1)}}{(2a_0 - 1)(\alpha - 1) - M(\alpha)} \tag{274}$$

For $\alpha \to 1$ the relation (274) converges to $a_n = \frac{\Gamma(n-1)\alpha+1}{\Gamma(n\alpha+1)} (a_{n-1} - b_{n-1})$ with $a_1 = \frac{a_0 - b_0}{\Gamma(1+\alpha)}$.

7.3.4. A model accounting for the sex of the partners [166]

Alkahtani et al. [166] suggested a model

$$^{ABC}D_t^{\alpha} y(t) = ay^2(t) - by(t) - (1-p)v(t)y^2(t) = = -by(t) + [a - (1-p)v] y^2(t) \tag{275}$$

where the time-dependent function $v(t)$ is a selection function giving individuals possibilities to choose partners of the same sex, that is it allows to account for non-reproductive couples.

Regarding the equilibrium point, and assuming, for the sake of simplicity, that the right-hand side Eq.(275) is time independent, it follows that [166]

$$ay^2 - by - (1-p)\nu y^2 = 0 \Rightarrow y = 0, \;\; y = \frac{b}{1-(1-p)\nu}, \; a \neq (1-p)\nu \tag{276}$$

Hence, the real equilibrium point will be when the rate of death with the difference between the birth contribution and the factor accounting for the partner sex becomes zero [166]. Moreover, if the inequality $a-(1-p)\nu < 0$ holds, then the living individuals will become extinct. However, when $a = (1-p)$, then the population will quickly vanish if the number of individuals is not quite enough to reproduce themselves and survive. Probably if the number of individuals is too big (beyond a certain threshold number of the population members), it could survive, thus suppressing, to some extent, the negative effect of the selection function $\nu(t)$ on the biological growth [166].

A numerical solution by the forward-corrector method was performed by presenting the model (275) in an equivalent Volterra version, namely

$$\begin{aligned} y(t) - y(0) &= \frac{1-\alpha}{M(\alpha)}\left[ay^2(t) - by(t) - (1-p)\nu(t)y^2(t)\right] + \\ &+ \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{\alpha-1}}\left[ay^2(st) - by(s) - (1-p)\nu(s)y^2(s)\right] ds \end{aligned} \tag{277}$$

Here, we can see the application of the ABC integral (see Remark 15) as well as [167]), where such an expansion of the model was related to the fading memory concept (a form close to the Volterra approach as discussed in Sections 6.2). Thus, this solution approach could be, to some extent, related to method used by Hassan et al. in [164] (see the preceding Section 7.3.1). More details about the numerical approximations applied are available in [166], and we will skip this point.

7.4. Logistic model with the Prabhakar fractional operator

7.4.1. Prabhakar function

The three-parameter Mittag-Leffler function is defined as [170]

$$E^{\gamma}_{\alpha,\beta} = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)\, n!} \tag{278}$$

For $\alpha = \beta = \gamma = 1$ we recover the exponential function. And, $E^{\gamma}_{\alpha,\beta}$ is an entire function. The Prabhakar kernel is defined as

$$e^{\gamma}_{\alpha,\beta}(\lambda; t) = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha}) \tag{279}$$

Thus, the Prabhakar fractional integral with a lower terminal 0 is defined as

$${}^{P}I^{\gamma}_{\alpha,\beta,\lambda}[f(t)] = \int_0^t e^{\gamma}_{\alpha,\beta}(\lambda; t-s) f(s)\, ds, \quad \alpha \in (0,1) \tag{280}$$

For $f(t) \in L^1(0,1)$ we have a relation to the Riemann-Liouville fractional integral as

$${}^{P}I^{\gamma}_{\alpha,\beta,\lambda}[f(t)] = \sum_{n=0}^{\infty} \frac{(\gamma)_n \lambda^n}{n!} {}^{RL}I_t^{\alpha n+\beta} f(t) \tag{281}$$

Moreover, ${}^{P}I^{\gamma}_{\alpha,\beta,\lambda}$ is linear and bounded from $L^p(0,1)$ into $L^p(0,1)$ for any $1 < p < \infty$ [154]. ${}^{P}I^{\gamma}_{\alpha,\beta,\lambda}$ allows definitions of two fractional derivatives [158]:

Prabhakar fractional derivative in Riemann-Liouville sense

$${}^{P}_{RL}D_{\alpha,\beta,\lambda} f(t) = \frac{d}{dt}\, {}^{P}I^{-\gamma}_{\alpha,1-\beta,\lambda} f(t) \tag{282}$$

Prabhakar fractional derivative in Caputo sense

$${}^{P}_{C}D_{\alpha,\beta,\lambda} f(t) = {}^{P}I^{-\gamma}_{\alpha,1-\beta,\lambda} \frac{df(t)}{dt} \tag{283}$$

The Laplace transform of ${}_{C}^{P}D_{\alpha,\beta,\lambda}f(t)$ when $f(t) = t^n$ is [154, 158] is

$$\mathcal{L}\left[{}_{C}^{P}D_{\alpha,\beta,\lambda}t^n\right](s) = s^{\beta-\alpha\gamma}(s^\alpha - \lambda)^\gamma \frac{\Gamma(n+1)}{s^{n+1}} \tag{284}$$

7.4.2. Prabhakar fractional logistic equation

After these preliminaries, let us consider the Prabhakar fractional logistic equation constructed in [154]

$$\Lambda(\alpha,\beta,\gamma,\lambda)\,{}_{C}^{P}D_{\alpha,\beta,\lambda}^{\gamma}y(t) = y(t)(1-y(t)) \tag{285}$$

where the factor $\Lambda(\alpha,\beta,\gamma,\lambda)$ is defined as: $\Lambda(\alpha,\beta,\gamma,\lambda) = 1 - \frac{\lambda}{1-\lambda}$, for $\alpha = 1$ and $\Lambda(\alpha,\beta,\gamma,\lambda) = \left[\frac{M(\alpha)}{1-\alpha}\right]^{\frac{(1-\alpha)}{\alpha}\gamma\lambda}$ when , $\alpha \neq 1$.

With the formal power series $y(t) = \sum_{n=0}^{\infty} a_n t^n$, suggested as a solution, and leading to $y(t)(1-y(t)) = \sum_{n=0}^{\infty}(a_n - b_n)t^{n\xi}$, with $b_n = \sum_{j=0}^{n} a_j a_{n-j}$, as many times demonstrated here, and applying the Laplace transform to both sides of Eq.(285), we have

$$\Lambda(\alpha,\beta,\gamma,\lambda)\,s^{\beta-\alpha\gamma}(s^\alpha-\lambda)^\gamma \sum_{n=0}^{\infty} a_n \frac{\Gamma(\xi n+1)}{s^{\xi n+1}} = \sum_{n=0}^{\infty}(a_n - b_n)\frac{\Gamma(\xi n+1)}{s^{\xi n+1}} \tag{286}$$

Further, we skip this point; the recurrence relation about the coefficients $a_{n+1} = R(a_n, \alpha, \beta, \gamma, \lambda, n)$ is already demonstrated, and the reader can develop it himself.

Remark 16 (On the model constructions and solutions developed: a standpoint)**.** *As mentioned at the beginning of this section, the models considered are obtained by replacing the time derivative in the Verhulst model with various fractional derivatives. There are no derivations based on some first modeling principles. Because of that, they are considered surrogate models, replacing but not the same because of the missing starting concepts of their build-ups.*

Furthermore, as we see, in each particular case, when a certain fractional derivative is used by replacement, the authors change (modify) the basic birth-death function on the right-hand side to accommodate the model to the solution they desire to develop because each type of derivative imposes new conditions that should be taken into account; however, these are not conditions imposed by the physics behind the model equation. As a result, we have different models providing different solutions that are not ready to compare. Nevertheless, the solutions developed reveal a wide range of techniques allowing for solving them but we did not compare the efficiencies and exactness of the final solutions since neither empirical data nor benchmarking solutions are available in the literature; such a trend in the model brush-up draws a new study direction beyond the horizon of this work.

However, it is noteworthy to see that the solutions suggested as formal power series [140–142, 150, 154] lead to efficient recursive relationships. This amazing result, tracing it back to the classic series solution of integer-order differential equations, is quite efficient because all fractional operators (derivative) in the considered examples can differentiate power-law functions (see, for instance the analysis in [168]. In this context, it might be expected that this approach would also be efficient when hereditary models involving fractional operators are constructed, avoiding derivative replacements.

Remark 17 (The forgotten inflection point)**.** *We can see from the analyzed solutions that the problem of the inflection points of the sigmoid-like solutions developed by different technologies was forgotten. Since the solutions are primarily power series with recurrences in the coefficient determination rather than continuously differentiable functions, this can be partially understood and explained. For example, if we consider the formal power series solution $y(t) = \sum_{0}^{\infty} a_n t^n$, the second-order derivative condition*

$\frac{d^2y(t)}{dt^2}=0$, *defining the inflection point, is*

$$\begin{aligned}&\frac{d^2y(t)}{dt^2}=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}=2a_2+6a_3t^2+a_nn(n-1)t^{n-2}=0\Rightarrow\\&\Rightarrow 2a_2+\sum_{n=3}^{\infty}a_nn(n-1)t^{n-2}=0\end{aligned} \tag{287}$$

Hence, it is obvious that we cannot define the inflexion point if the solution is as a power series. For the other solutions developed, similar tests can be done too.

Remark 18 (An open problem in fractional logistic modeling)**.** *As commented above and from the analyzes of the models discussed, it is evident that real data fitting by the solution developed is practically missing, which, to a greater extent, makes the validations and comparative analyzes difficult to carry out. In this context, the estimation of the fractional order in the models, using experimental data, as a serious inverse problem, is an open problem still undeveloped. Especially for the cases of the fractional logistic model, for instance, we refer to [169], where a Bayesian approach was applied. Hence, there is a new area where efforts might be directed to allow real validation of what in fractional population dynamics was developed as models.*

8. Non-local operators related to the sigmoid function

8.1. Fractional derivative and integrals related to sigmoid function implementation

8.1.1. From the Riemann-Liouville fractional derivative with respect to another function to the Hilfer construction

If the function $\psi(u)>0$ is monotonically increasing on $[a,b]$ and its derivative $d\psi/du>0$ is continuous on (a,b), where $a,b\in\mathbb{R}$, then with $\alpha>0$ we may represent the Hilfer fractional derivative of order α, $0<\alpha<1$ and type β, $0\le\beta\le1$ concerning x as [171, 172]

$$D_{a^+}^{\alpha,\beta}f(x)={}^{1-\alpha}J_{a^+}^{\beta}\frac{d}{dx}\left({}^{1-\alpha}J_{a^+}^{1-\beta}f(x)\right) \tag{288}$$

if the right-hand side of Eq.(288) exists.

The Hilfer definition, Eq. (288), allows generalization of the so-called ψ fractional integral and derivative, namely

$${}^{H}D_{a^+}^{\alpha,\beta;\psi}f(x)=I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\frac{d\psi}{dx}}\frac{d}{dx}\right)^{n}I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(x) \tag{289}$$

$${}^{H}D_{b^-}^{\alpha,\beta;\psi}f(x)=I_{b^-}^{\beta(n-\alpha);\psi}\left(\frac{1}{\frac{d\psi}{dx}}\frac{d}{dx}\right)^{n}I_{b^-}^{(1-\beta)(n-\alpha);\psi}f(x) \tag{290}$$

where $n=[\alpha]+1$. Then, the corresponding fractional integrals are

$$I_{a^+}^{\alpha;\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{d\psi(t)}{dt}\frac{f(t)}{(\psi(x)-\psi(t))^{1-\alpha}} \tag{291}$$

$$I_{b^-}^{\alpha;\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{d\psi(t)}{dt}\frac{f(t)}{(\psi(x)-\psi(t))^{1-\alpha}} \tag{292}$$

Napoles-Valdes suggested the *k-generalization* of the $\psi-Hilfer$ integral as [172]

$$I_{G,a^+}^{\alpha,k;\psi}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x\frac{d\psi(t)}{dt}\frac{f(t)}{G\left((\psi(x)-\psi(t)),\frac{\alpha}{k}\right)},\quad k>0 \tag{293}$$

$$I_{G,b^-}^{\alpha,k;\psi}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_n^b\frac{d\psi(t)}{dt}\frac{f(t)}{G\left((\psi(x)-\psi(t)),\frac{\alpha}{k}\right)},\quad k>0 \tag{294}$$

where $G(z,t) \in AC[a_1, a_2]$.
The corresponding $\psi - Hilfer$ fractional derivatives are

$$ {}^{H}D_{G,a^+}^{\alpha,k;\psi} f(x) = I_{G,a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\frac{d\psi}{dx}} \frac{d}{dx} \right)^n I_{G,a^+}^{(1-\beta)(n-\alpha);\psi} f(x), \quad k>0, \quad n=[\alpha]+1 \tag{295} $$

$$ {}^{H}D_{G,b^-}^{\alpha,k;\psi} f(x) = I_{G,a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\frac{d\psi}{dx}} \frac{d}{dx} \right)^n I_{G,b^-}^{(1-\beta)(n-\alpha);\psi} f(x), \quad k>0, \quad n=[\alpha]+1 \tag{296} $$

From these general definitions, we may see the following reductions (specific definitions) [172], for instance:

- For $k=1$ and $G(z,\alpha) = z^{1-\alpha}$ as well as $d\psi(t)/dt = 1$ we get the fractional Riemann-Liouville operator
- For $k=1$ and $G(z,\alpha) = z^{1-\alpha}$, but $d\psi(t)/dt = 1/t$, then we get the Hadamar fractional operator (see [171].

More such examples are available in [172].
At the end of this section, we refer to a particular case [172] that can be useful in the rest of this article, namely:
Using the definition of the sigmoid $\sigma(x) = \frac{1}{1+e^{-x}}$ and its derivative $d\sigma(x)/dx = \sigma(x)(1-\sigma(x))$, and with $k=1$, $G(z,\alpha) = z^{1-\alpha}$, as well as with $\psi(u) = \sigma(u)$ the following fractional integral can be constructed

$$ I_{a^+}^{\alpha;\sigma} f(x) = \int_a^x \frac{\sigma(t)(1-\sigma(t))}{(\sigma(x)-\sigma(t))^{1-\alpha}} f(t)\,dt \tag{297} $$

and the corresponding derivative is

$$ {}^{H}D_{a^+}^{\alpha,\beta} f(x) = I_{a^+}^{\beta(n-\alpha);\sigma} f(x) = \left(\frac{1}{\frac{d\sigma(x)}{dx}} \frac{d}{dx} \right)^n I_{a^+}^{(1-\beta)(n-\alpha);\sigma} f(x) \tag{298} $$

In the context of applications of the sigmoid function, Napoles-Valdes [172] suggested, but not developed, a generalized version as

$$ \sigma(x) = \frac{1}{1+e^{-g(x)}} \tag{299} $$

with a smooth function $g(x)$. We will see how this idea is developed further in Section 8.2.

8.2. ψ-Hilfer fractional derivative: a specific case

If we define $\psi(x) = \sigma(x) = \frac{1}{1-e^{-x}}$ [173], as suggested in [172], and using the definitions of left and right fractional integrals (see Eq.(291) and Eq.(292)), and concerning an increasing and positive monotone function on the real axis, we may construct [173]

$$ \begin{aligned} I_{a^+}^{\alpha,\sigma} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{e^{-t}}{(1+e^{-t})^2} \left(\frac{1}{1+e^{-x}} - \frac{1}{1+e^{-t}} \right)^{\alpha-1} f(t)\,dt = \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x \sigma(t)(1-\sigma(t))(\sigma(x)-\sigma(t))^{\alpha-1} f(t)\,dt \end{aligned} \tag{300} $$

assuming that $\frac{d\psi(x)}{dx}$ is continuous on the interval (a,b).
Now, let us recall that the definitions of the left and $\psi - Hilfer$ derivatives are Eq.(289) and Eq.(290). Then, using these constructions, fractional derivatives with a sigmoid function as a kernel can be defined as [173]:

$$ {}^{H}D_{a+}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma((1-\beta)(n-\alpha))} \int_a^t \sigma(\tau)(1-\sigma(\tau))(\sigma(x)-\sigma(\tau))^{1-\alpha} f(\tau)\,dt d\tau \tag{301} $$

From this general definition, the following special cases can be developed, namely.

- For $\beta \to 0$,$n = 1$

$$ {}^{H}D_{a+}^{\alpha,0;\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \sigma(\tau)(1-\sigma(\tau))(\sigma(x)-\sigma(\tau))^{1-\alpha} f(\tau)\, dt d\tau \tag{302} $$

- For $\beta \to 1$,$n = 1$

$$ {}^{H}D_{a+}^{\alpha,0;\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \frac{1}{(\sigma(x)-\sigma(\tau))^{\alpha}} \frac{d}{dt} \int_{a}^{t} \sigma(\tau)(1-\sigma(\tau)) f(\tau)\, dt d\tau \tag{303} $$

- For $\beta \to 0$,$n \neq 1$

$$ \begin{aligned} {}^{H}D_{a+}^{\alpha,0;\psi} f(x) &= \frac{1}{\Gamma^{2}(n-\alpha)} \int_{a}^{t} (\sigma(\tau)(1-\sigma(\tau)))^{1-n} \times \\ &\times \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} \sigma(\tau)(1-\sigma(\tau))(\sigma(x)-\sigma(\tau))^{n-\alpha} f(\tau)\, dt d\tau \end{aligned} \tag{304} $$

- For $\beta \to 1$,$n \neq 1$

$$ \begin{aligned} {}^{H}D_{a+}^{\alpha,0;\psi} f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (\sigma(\tau)(1-\sigma(\tau)))^{1-n}(\sigma(x)-\sigma(\tau))^{n-\alpha-1} \times \\ &\times \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} \sigma(\tau)(1-\sigma(\tau)) f(\tau)\, dt d\tau \end{aligned} \tag{305} $$

8.2.1. Some properties

Based on the boundedness of the sigmoid kernel, Eq.(301) defines a bounded operator for all $n-1 < \alpha < n$ and $0 \leq \beta \leq 1$, that is [173]

$$ \left\| {}^{H}D_{o^{+}}^{\alpha,\beta} \right\|_{C,\gamma,\sigma} \leq K \|f\|_{C,\gamma,\sigma}, \quad K = \left(\frac{1-e^{-b}}{2\left(1+e^{-b}\right)}\right)^{n-\alpha} \frac{1}{\Gamma(n-\gamma+1)\Gamma(\gamma-\alpha+1)} \tag{306} $$

Then, from ${}^{H}D_{a^{+}}^{\alpha,\beta} f(x) = I_{a^{+}}^{\gamma-\alpha;\sigma} D_{a^{+}}^{\gamma;\sigma} f(x) = I_{a^{+}}^{\gamma-\alpha;\sigma} D_{a^{+}}^{\gamma;\sigma} (\sigma(x)-\sigma(a))^{\delta-1}$ (from Lemma 1 and Lemma 2 in [173]). Further, Liu et al. [173] suggested that the simple sigmoid function defined by Eq.(300) can be controlled by a parameter α such that

$$ \sigma_{\alpha}(x) = \frac{1}{1+e^{-\alpha x}} \Rightarrow \frac{d\sigma_{\alpha}(x)}{dx} = \alpha \frac{e^{-\alpha x}}{(1+e^{-\alpha x})^{2}} = \alpha \sigma_{\alpha}(x)(1-\sigma_{\alpha}(x)) \tag{307} $$

That is, this is a particular version of the idea of Napoles-Valdes presented byEq. (299) with $g(x) = \alpha$, $0 < \alpha < 1$. Then, a fractional operator on this basis was formulated as [173]

$$ \begin{aligned} {}^{H}D_{a^{+}}^{\alpha,\beta} f(x) &= \frac{1}{\Gamma(\beta(n-\alpha))} \frac{1}{\alpha^{n-2}} \int_{a}^{x} \sigma(t)(1-\sigma(t))(\sigma(x)-\sigma(t))^{\beta(n-\alpha)-1} \left(\frac{1}{\sigma(t)(1-\sigma(t))} \frac{d}{dt}\right)^{n} \times \\ &\times \frac{1}{\Gamma((1-\beta)(n-\alpha))} \int_{a}^{t} \sigma(\tau)(1-\sigma(\tau))(\sigma(x)-\sigma(\tau))^{(1-\beta)(n-1)} f(\tau)\, dt d\tau \end{aligned} \tag{308} $$

Applying this derivative to a simple population growth model [173] as

$$ \left({}^{H}D_{a^{+}}^{\alpha,\beta} f\right)(t) = s_{0} \exp(\lambda t) \tag{309} $$

$$f(t) = s_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\mu - \chi + 1)} \left(\frac{1}{1+e^{-t}} - \frac{1}{1+e^{-\alpha}} \right)^{k\mu - \chi}, \quad \mu = n(1-\beta) + \alpha\beta, \quad \chi = (1-\beta)(n-\alpha) \tag{310}$$

For $\lambda > 0$ and $n-1 < \alpha < n$, and $0 \leq \beta \leq 1$ the result can be expressed through the one-parameter Mittag-Leffler function as [173]

$$f(t) = s_0 E_\alpha \left(\lambda \left(\left(\frac{1}{1+e^{-t}} - \frac{1}{1+e^{-\alpha}} \right)^{\alpha} \right) \right) \tag{311}$$

Similarly, versions of the Korteweg-deVries equation, Burger's equation, the shallow water model, and the reaction-diffusion equation were suggested by simple replacements of the time derivatives in the original models by ${}^{H}D_{a^+}^{\alpha,\beta} f(x,t)$; however, no solutions and consequent analyses were provided.

8.3. Fractional derivative based on hyperbolic secant as a kernel

A fractional, Caputo-type, and left-sided derivative with a hyperbolic secant-based kernel was proposed by Rezapour et al. [174] (with $0 < \alpha < 1$, $f(t) \in H^1(a,b)$, $t > a$)

$$^{\sigma}D_a^{\alpha} f(t) = C_1(\alpha) \int_a^t \operatorname{sech}^2 \left(\frac{s-t}{1-\alpha} \right) \left\{ \frac{df(s)}{ds} \right\} ds \tag{312}$$

where the term $\left\{ \frac{df(s)}{ds} \right\}$ denotes a H^1 a distributional derivative as defined in general as [175]

$$\int T'(t)\phi(t)\,dt = -\int T(t)\phi(t)\,dt \tag{313}$$

The definition (312) reduces to H^1 derivative for $\alpha \to 1$. Moreover, if $f(t)$ is at least twice differentiable, then ${}^{\sigma}D_a^{\alpha} f(t)$ is differentiable, too, and if $(df/dt\,)_{t=0} = 0$ we have

$$^{\sigma}D_a^{\alpha} f(t) \left[{}^{\sigma}D_a^{1} f(t) \right] = {}^{\sigma}D_a^{1} \left[{}^{\sigma}D_a^{\alpha} f(t) \right], \quad 0 < \alpha < 1 \tag{314}$$

It is important to mention that ${}^{\sigma}D_a^{\alpha} f(t)$ does not satisfy the memory principle (see Remark 19) but can be approximated by a corresponding fractional derivative with a lower terminal $t - L$ (see the complete exposition of this theorem in [174]).

Remark 19 (The memory principle for fractional derivative)**.** *If we have a fractional operator of Caputo-type [174]*

$$D_K^{\alpha} f(t) \equiv \int_a^t K(t-s) \frac{df(s)}{ds} ds, \quad f \in H^1(a,b) \tag{315}$$

Then the memory principle describes the history of $f(t)$at the proximity of the lower terminal, i.e. for $t \to a$. In such a situation, if the memory length (distance) is defined as Lso that $a + L < t + b$ the errors in approximation of the fractional operator are defined as

$$Er_{L,a,\alpha}(t) = \left| {}_{a}D_K^{\alpha} f(t) - {}_{t-L}D_K^{\alpha} f(t) \right| \tag{316}$$

If $df/dt\ < M$ for $t \in (a,b)$ and with $0 < \alpha < 1$, then, for instance, in the case of the Caputo derivative, the error is

$$Er_{L,a,\alpha}(t) = \left| \frac{1}{\Gamma(1-\alpha)} \int_{t-L}^t \frac{1}{(t-s)\alpha} \frac{df(s)}{ds} ds \right| \leq M \frac{L^{1-\alpha}}{|\Gamma(2-\alpha)|} \tag{317}$$

Now, if $Er_{L,a,\alpha}(t) \leq \varepsilon$, $\varepsilon > 0$ and $a + L \leq t \leq b$ the memory distance should be

$$L > \left(\frac{M}{\varepsilon |\Gamma(2-\alpha)|} \right)^{\frac{1}{\alpha-1}} \tag{318}$$

Remark 20. *We can see that these are only initial attempts to construct fractional operators, precisely to demonstrate that the $\psi - Hilfer$ derivative works with the sigmoid function. The works of Napoles-Valdes [172] and Liu et al. [173] were published around the same time at the end of 2020, and they provide nearly similar results; however, the formulation in [172] is more fundamentally rigorous. The fractional* *operator with a hyperbolic secant as a kernel is still at the beginning level. The literature browsing, however, did not detect any development of these operators.*

9. Discrete fractional-difference logistic equations

9.1. Some definitions

Let us now consider the discrete logistic equation (map) conceived by May [176,177] (see further Eq.(343) in Section 10)

$$x_{n+1} = Kx_n (1 - x_n) \tag{319}$$

demonstrating chaotic behavior for K ranging from 3.57 and 4.

Following Edelman [177], the simplest construction of a discrete fractional equation is

$$\Delta^{\alpha}{}_{a,h} x(t) = f(x) \tag{320}$$

where the discrete forward h-difference operator acts as $\Delta_h f(x) = f(x+h) - f(x)$, and commonly its step h is accepted $h = 1$.

Since working in the area of fractional operators, we have dealt with integral convolution integral, which are, in fact, summation operators, thus jumping into the field of discrete models, we have summation-difference operators with an initial point a (commonly accepted as $a = 0$). It is important to stress the attention on the fact that, despite the term fractional, when $0 < \alpha \leq 1$, really the operators are defined for any real $\alpha \in \mathbb{R}$ [177].

Hence, the discrete fractional difference Caputo-like equation is [177]

$$^{C}\Delta^{\alpha} x(n) = -G_K (n + \alpha - 1, x(n + \alpha - 1)), \quad 0 < \alpha < 1, \quad x(0) = x_0, \quad n \in \mathbb{N} \tag{321}$$

where $K$is a parameter (a coefficient, see more about it in Section 10).

Equation (321) can be presented as a fractional–difference map with failing factorial-law memory [177], namely

$$x_n = x_0 - \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-1} (n - s - 2 + \alpha)^{(\alpha-1)} G_K (s, x_s) \tag{322}$$

The equivalent form of Eq.(322) , through the Bernoulli numbers $B(n, \alpha; s)$, is [177,178]

$$x_n = x_0 - \sum_{s=0}^{n-1} B(n-1, \alpha; s) G_K (s, x(s)), \quad B(n, \alpha; s) = \begin{pmatrix} n - s + \alpha - 1 \\ n - s \end{pmatrix}, \quad 0 \leq s \leq n \tag{323}$$

Then, the failing factorial function is defined as

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha) t^{\alpha}}, \quad t \neq -1, -2, -3, \ldots \tag{324}$$

It is worth noting that the failing factorial-law memory function is asymptotically power-law memory [177], that is

$$\lim_{t\to\infty} \left(\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha) t^{\alpha}} \right) = 1, \quad \alpha \in \mathbb{R} \tag{325}$$

9.2. Fractional-difference logistic equation

Equation (319) was reconsidered as [179]

$$\Delta x_n = \mu x_n (1 - x_n) \Rightarrow \Delta x_{n+1} = (1 + \mu) x_n - \mu x_n^2, \quad x(0) = c \tag{326}$$

and formulated through the left Caputo-like difference in the form [179]

$$^{C}\Delta_a^{\alpha} x(t) = \mu x(t + \alpha - 1) (1 - x(t + \alpha - 1)), \quad \Delta^k x(a) = x_k \tag{327}$$

with $t \in \mathbb{N}_{a+1-\alpha}, x(a) = c$, and a difference order $\alpha \in (0, 1]$, where $^{C}\Delta_a^{\alpha} x(t)$ is the left Caputo-like difference and $\mathbb{N} = (a, a+1, a+2, \ldots)$ $a \in \mathbb{R}$ fixed .

Note: To clarify the above construction, recall (see the comments about Eq.(320)) that the action of the delta difference operator $\Delta_{h=1}$ on a function $f(n)$ is $\Delta f(n) = f(n+1) - f(n)$

With the Caputo-like delta difference for $x(t)$ defined on $\mathbb{N}_a$ [180]

$$ {}^{C}\Delta_a^{\alpha} x(t) := \Delta_a^{-(m-\alpha)} \Delta^m x(t) = \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-\sigma(s))^{(m-\alpha-1)} \Delta_s^m x(t) \tag{328}$$

for $0 < \alpha \notin \mathbb{N}$ and $t \in \mathbb{N}_{a+m-\alpha}$, $m = [\alpha] + 1$.

A fractional sum of order $0 < \alpha$, ($\mathbb{N}_a \to \mathbb{R}$) with a starting point a is defined by

$$ \Delta_a^{-\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} x(s), \quad \sigma(s) = s+1, \quad t \in \mathbb{N}_{a+\alpha} \tag{329}$$

where $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ is the *failing function*.

And, therefore,

$$ {}^{C}\Delta_a^{\alpha} x(t) := \Delta_a^{-(m-\alpha)} \Delta^m x(t) = \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-\sigma(s))^{(m-\alpha-1)} \Delta_s^m x(t) \tag{330}$$

for $0 < \nu \notin \mathbb{N}$ and $t \in \mathbb{N}_{a+m-\alpha}$, $m = [\alpha] + 1$.

Now, for Eq.(327) with $m = [\alpha]+1$ and $k = 0, \dots, m-1$, the equivalent discrete integral equation is [179]

$$ \begin{aligned} x(t) &= x_0(t) + \frac{1}{\Gamma(\alpha)} \sum_{s=a+m-\alpha}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)) \\ x_0(t) &= \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a) \end{aligned} \quad , \quad t \in \mathbb{N}_{a+m} \tag{331}$$

In Eq.(331) the discrete kernel function is $\frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)}$ and $(t-\sigma(s))^{(\alpha-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s+1-\alpha)}$.

Note: During the transition from Eq.(328) to Eq.(331) there is a change in the time domain from $t \in \mathbb{N}_{a+m-\alpha}$ to $t \in \mathbb{N}_{a+m}$, but the function $x(t)$ is preserved to define on the isolated time scale $t \in \mathbb{N}_a$, when the summation Eq.(328) is carried out [179].

9.3. Discrete fractional-difference logistic equation: Chaos

9.3.1. Chaos in fractional sine

Wu et al. [182] considered a version of the integer-order sine map $x_{n+1} = x_n + \mu \sin(x_n)$ (see more detailed analysis on this problem and its fractionalized counterparts in the next Section 10). With ${}^{C}\Delta_a^{\alpha} x(t)$, defined above, it was reformulated as

$$ {}^{C}\Delta_a^{\alpha} x(t) = \mu \sin(x(t+\alpha-1)), \quad 0 < \nu < 1 \tag{332}$$

Then, the two-dimensional standard map (as used by the authors)

$$ \begin{aligned} x_{n+1} &= x_n - K \sin(z_n) \\ z_{n+1} &= z_n + x_{n+1} \end{aligned} \tag{333}$$

was modified as

$$ \begin{aligned} &{}^{C}\Delta_a^{\alpha} x(t) = -K \sin(z(t+\alpha-1)) \\ &z_n = z_{n-1} + x_n \end{aligned} \quad , \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_{a+1-\alpha} \tag{334}$$

For Eq.(332), with the discrete kernel $(t-\sigma(s))^{(\nu-1)}$, the explicit numerical formula is [182]

$$ x_n = x(a) + \frac{\alpha}{\Gamma(\alpha)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \sin(x(j-1)) \tag{335}$$

Consequently for (334) the numerical version is

$$ \begin{aligned} x_n &= x(a) + \frac{\mu}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \sin(z(j-1)) \\ z_n &= z_{n-1} + x_n \end{aligned} \quad , \quad n \geq 1 \tag{336}$$

9.3.2. Chaos and synchronization

For $a = 0$ and $\alpha = 1$, Eq.(327) reduces to Eq.(326) [179]. Applying the transformation

$$x_n = \frac{1+\mu}{\mu}\alpha_n \tag{337}$$

Equation (326) can be transformed as

$$\alpha_{n+1} = (1+\mu)\left(\alpha_n - \alpha_n^2\right), \quad \alpha_0 = \frac{\mu}{1+\mu}u_0 \tag{338}$$

Note: In Eq.(337) the function $(1+\mu)/\mu$ is the simplest form of a sigmoidal transformation because $(1+\mu)/\mu = 1/(1+1/\mu) \to 1$ for $\mu \to \infty$ [181], so that for large values of μ we have $x_n \to \alpha_n$ and $\alpha_0 \to u_0$.

With Eq.(328) and Eq.(329), defined above, the following discrete integral form of Eq.(326) can be formulated [179]

$$x(t) = u_0 + \frac{\mu}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} x(t-s-1)(1-x(t-s-1)), \quad t \in \mathbb{N} \tag{339}$$

with a corresponding numerical scheme, using $(t-\sigma(s))^{(\alpha-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s+1-\alpha)}$, expressed as

$$x_n = x_0 + \frac{\mu}{\Gamma(\mu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} x(j-1)(1-x(j-1)) \tag{340}$$

Hence, Eq.(340) has a discrete kernel function $(t-\sigma(s))^{(\nu-1)}$ in contrast to the integer-order Eq.(326) and, therefore, x_n is a function (depends on) of its past values.

On the basis of the results commented on above, precisely Eq.(327) considered as *a master system*. Wu and Baleanu [183] suggested *a coupled slave logistic map* (a term used by the authors), namely

$$^{C}\Delta_a^{\alpha} z(t) = \mu z(t+\alpha-1)(1-z(t+\alpha-1)) + Hx(t+\alpha-1), \quad x(t+\alpha-1), \quad x(a) = x_0 \tag{341}$$

So, for a long time we have $|x(t) - z(t)|_{t\to\infty} \to 0$. Then, Eq.(327) and Eq.(341) form a synchronized system, namely

$$\begin{aligned} x(t) &= u_0 + \frac{\mu}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} x(t-s-1)(1-x(t-s-1)) \\ z(t) &= z(a) + \frac{\mu}{\Gamma(\alpha)} \sum_{s=a+1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[z(t-s-1)(1-z(t-s-1)) + H\right], \quad t \in \mathbb{N}_{a=1} \\ H &= Hx(t-s-1)\, z(t-s-1) \end{aligned} \tag{342}$$

Remark 21 (Some comments on the terminology used)**.** *Here, we refer to the term fractional discrete map used in [179, 182, 183]. As commented by Edelman (Section 3.1 in [177]) they are the only solutions of the discrete fractional-difference logistic equation, but not logistic maps. However, the results in [179, 182, 183] show chaotic behaviors with bifurcations clearly demonstrated and visualized. At this point, we hope the reader may see the differences in the positions of these authors and that of Edelman [177] (see the next Section 10) and take the right answer to the emerging terminology discrepancies.*

10. Logistic maps

The regular logistic map used in population dynamics, economics, and descriptions of other dynamic processes is simply defined as [176, 177, 191]

$$x_{n+1} = Kx_n(1-x_n) \tag{343}$$

and its simplest generalization is [184, 191]

$$x_{n+1} = Kx_n^p\left(1-x_n^q\right), \quad x \in [0,1], \quad p, q > 0 \tag{344}$$

For more versions see [191] and the references therein.

In this section, we credit the results of Zaslavsky, Tarasov, and Edelman [185–190], which form a solid background to understand the creation of logistic maps and their fractionalized versions.

10.1. Standard map

Following Tarasov and Zaslavsky [185], the mapping $x_{n+1} = f(x_n)$ is memoryless because x_{n+1} depends only on x_n. To construct a map with memory, we need to connect the discrete value x_{n+1} to its preceding values $x_n, x_{n-1}, \ldots, x_1$. This functionally can be expressed as $x_{n+1} = \sum_{k=1}^{n} f(x_k)$where $f(x)$ is a function describing a discrete map. Since the number of the preceding values is limited (finite), the memory described in this way is not complete as we desire it to be [185]. The map with long-time memory is defined as [185] $x_{n+1} = \sum_{k=1}^{n} V_\alpha(n,k) f(x_k)$, where $V_\alpha(n,k)$ are weighting functions and they should be determined through solutions of differential equations, as explained further in Remarks 22 and 23.

Let us consider, for instance, the equation of motion perturbed by a periodic sequence of kicks (delta function-type of pulses) with amplitude K [187, 188]

$$\frac{d^2x}{dt^2} + K\sin(x)\sum_{n=0}^{\infty}\delta\left(\frac{t}{T} - n\right) = 0 \tag{345}$$

With a period $T = 1$ (for the sake of simplicity) we read

$$\frac{d^2x}{dt^2} + K\sin(x)\sum_{n=0}^{\infty}\delta(t-n) = 0 \tag{346}$$

The transformation into a Hamiltonian form by $dx/dt = p$, yields

$$dx/dt = p, \quad \frac{dp}{dt} + K\sin(x)\sum_{n=0}^{\infty}\delta(t-n) = 0 \tag{347}$$

and allows (347) to be presented as a discrete map [187, 188]

$$x_{n+1} = x_x + p_{n+1}, \quad p_{n+1} = p_n - K\sin(x_n) \tag{348}$$

This is the *universal map* and it can be generalized as [185]

$$x_{n+1} = x_x + p_{n+1}, \quad p_{n+1} = p_n - G(x_n) \tag{349}$$

With $G(x) = \sin(x)$ we get the *Chirikov-Taylor map* [185], also known as a standard map [187] or *Chirikov map* [193].

Remark 22 (Cauchy-type problem and universal map equivalence)**.** *The Cauchy-type problem [186] (in the original notations)*

$$\frac{dx(t)}{dt} = p(t), \quad \frac{dp(t)}{dt} = -KG[x_k]\sum_{k=1}^{\infty}\delta\left(\frac{t}{T} - k\right), \quad x(0) = x_0, \quad p(0) = p_0 \tag{350}$$

is equivalent to a universal map with a period T, expressed as

$$x_{n+1} = x_0 + p_0(n+1)T - kT^2\sum_{k=1}^{n} G[x_k](n+1-k), \quad p_{n+1} = p_0 - KT\sum_{k=0}^{n} G[x_k] \tag{351}$$

Further, if (350) is presented as

$$\frac{d^2x(t)}{dt^2} = G[t, x(t)], \quad 0 \le t \le t_{final}, \quad x(0) = x_0, \quad \frac{dx(0)}{dt} = p(0) \tag{352}$$

It is equivalent to the Volterra equation of the second kind [186]

$$x(t) = x_0 + p_0 t + \int_0^t G[s, x(s)](t-s)\,ds \tag{353}$$

With $G[t, x(t)] = KG[x_k]\sum_{k=1}^{\infty}\delta\left(\frac{t}{T} - k\right)$ we get (350).

Remark 23 (On the equivalent Volterra equations of second kind)**.** *We saw that in many cases discussed here, there are equivalent models as Volterra equations of the second kind. Thus, we can see the common ground of memory in both continuous and discrete models. Precisely, working with discrete models and recalling the above comments about the functionality $x_{n+1} = \sum_{k=1}^{n} f(x_k)$, where the past values are weighted by $V_\alpha(n,k)$, we can look back at the models in Section 5, and especially in Section 6, where the weighting of the past values is by the correlation (memory) function. In all cases, discrete and continuous, the summation is a performance of memory. Referring to the beginning paragraph of Section 10.1, we can see that the incomplete memory due to the truncated series of past values has its analog in the convolution integrals when the memory functions can affect (weights) only the closest values rather than all in the past.In cases, discrete and continuous, the starting point (the point $t = 0$) is the moment where the process starts and is counted during the entire process duration (not the chronological time): Recall the comments in Section 6.4, precisely referring to the concept expressed by Hilfer [117, 118].*

10.1.1. Dissipative standard map

The dissipative standard map is defined as

$$X_{n+1} = X_n + \mu Y_{n+1} + \Omega, \quad Y_{n+1} = e^{-q}\left(Y_n + \varepsilon \sin\left(X_n\right)\right), \quad \mu = \frac{\left(e^{-q} - 1\right)}{q} \tag{354}$$

and also known as the Zaslavsky map [187]. The shift Ω can be assumed as zero since it does not play a significant role in the map dynamics [187], and then

$$X_{n+1} = X_n + P_{n+1}, \quad P_{n+1} = -bP_n - Z\sin\left(X_n\right), \quad Z = -\varepsilon\mu e^{-q}, \quad P_n = \mu Y_n, \quad b = -e^{-q} \tag{355}$$

For $b = -1$ and $Z = K$ from the map (355) we recover the standard map (348). When $q \to \infty$ and $b \to 0$ as well as with $Z = -K$, the map (355) reduces to the one-dimensional Arnold's map [187, 194].

$$X_{n+1} = X_n + K\sin\left(X_n\right) \tag{356}$$

10.2. Fractional map

10.2.1. Fractional standard map with dissipation

A fractional generalization of Eq.(345) was conceived by Nigmatulin [195] as a discrete map corresponding to a fractional equation of order $0 < \alpha \le 2$ and as a generalization of the standard map, *via* a Riemann-Liouville derivative ${}^{RL}D_t^{\alpha}x$, namely

$${}^{RL}D_t^{\alpha}x + K\sin(x)\sum_{n=0}^{\infty}\delta(t-n) = 0, \quad {}^{RL}D_t^{\alpha}x = \frac{1}{\Gamma(2-\alpha)}\frac{d}{dx^2}\int_0^t \frac{x(s)}{(t-s)^{\alpha}}ds, \quad 1 < \alpha \le 2 \tag{357}$$

Through the momentum $p(t) = {}^{RL}D_t^{\alpha-1}x(t)$ and the initial conditions ${}^{RL}D_t^{\alpha-1}x(0+) = p_1$, [187] ${}^{RL}D_t^{\alpha-2}x(0+) = b$, the fractional standard map can be built [187, 188]

$$x_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n} p_{k+1}V_\alpha(n-k+1), \quad 1 < \alpha \le 2, \quad p_{n+1} = -bp_n - Z\sin(x_n) \tag{358}$$

where $Z = -\varepsilon\mu e^{-q}$, $P_n = \mu Y_n$, $b = -e^{-q}$ (see Eq.(355)). As mentioned in [187] for $b = -1$ and $Z = K$, then the map (358) reduces to a fractional dissipative standard map with $T = 1$. The map (358) is not a result of a fractional differential equation but from $p_n \to -bp_n$ in the fractional standard map [187].

10.2.2. A map with a fractional derivative in the kicked term

The equation of a kicked damped rotator is defined as [187, 188]

$$\frac{d^2x}{dt^2} - q\frac{dx}{dt} - K\sin(x)\sum_{n=0}^{\infty}\delta(t-n) = 0 \tag{359}$$

It can be fractionalized by applying the Caputo derivative ${}^{C}D_t^{\alpha}x(t)$ (see Eq.(361)), namely

$$\frac{d^2X(t)}{dt^2} - q\frac{dX(t)}{dt} - \varepsilon^C\sin\left({}^{C}D_t^{\beta}X\right)\sum_{n=0}^{\infty}\delta(t-n) = 0, \quad 0 \le \beta < 1 \tag{360}$$

$$ {}^{C}D_t^{\beta}X = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{1}{(t-s)^{\beta}} \frac{dX(s)}{ds} ds \tag{361} $$

This is *the first fractional dissipative standard map* [187, 188].

The fractional equation (359) is equivalent to the discrete map [187, 188]

$$ X_{n+1} = X_n + \frac{1-e^{-q}}{q} Y_{n+1}, \quad Y_{n+1} = e^q \left\{ Y_n + \varepsilon \sin \left[\frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{n-1} Y_{k+1} W_{2-\beta}(q, k-n) \right] \right\} \tag{362} $$

where the function is defined through the incomplete gamma function $\Gamma(a, b)$

$$ W_{2-\beta}(q, k-n) = a^{\beta-1} e^{a(b+1)} \left[\Gamma(1-\beta, ab) - \Gamma(1-\beta, a(b+1)) \right], \quad \Gamma(a, b) = \int_b^{\infty} y^{a-1} e^{-y} dy \tag{363} $$

10.2.3. A map with a fractional derivative in the unkicked term

If the fractionalization Eq.(359) is done by replacements of both integer-order time derivatives (in the left-hand side) by Riemann-Liouville fractional derivatives of different fractional orders, then

$$ {}^{RL}D_t^{\alpha} X(t) - q\, {}^{RL}D_t^{\beta} X(t) - \varepsilon \sin(X) \sum_{n=0}^{\infty} \delta(t-n) = 0, \quad 1 < \alpha \le 2, \quad 0 \le \beta < 1, \quad q \in R \tag{364} $$

and it is equivalent to the following discrete map [187, 188]

$$ X_{n+1} = \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n} Y_{k+1} W_{\alpha}(q, k-n-1), \quad Y_{n+1} = e^q \left[Y_n + \varepsilon \sin(X_n) \right] \tag{365} $$

with $W_{\alpha}(q, k-n-1) = a^{1-\alpha} e^{a(b+1)} \left[\Gamma(\alpha-1, ab) - \Gamma(\alpha-1, a(b+1)) \right]$. This can be considered as a fractional generalization of the dissipative standard map (348) with $\Omega = 0$ [187]. For $\alpha = 2$, the map (365) reduces to the discrete map (348).

With the variables $P_n = \mu Y_n$, $b = -e^q$ and $Z = -\mu\varepsilon e^q$ we get a new form of the map (365), namely

$$ X_{n+1} = \frac{1}{\mu} \frac{1}{\Gamma(\alpha-1)} \sum_{k=0}^{n} P_{k+1} W_{\alpha}(q, k-n-1), \quad P_{n+1} = -bP_n - Z \sin(X_n) \tag{366} $$

Note: In Eq.(359), Eq. (360) and Eq.(364), the second (damping) terms are with negative signs because the coefficients q could be positive or negative [187, 188].

10.2.4. Fractionalized standard map: Riemann-Liouville or Caputo derivative?

Here we refer to [189] where a comparative analysis of the fractional standard map with the two most used derivatives with power-law kernels was done. In more detail, considering the initial condition problem, we have a standard map [189]

$$ p_{n+1} = p_n - K \sin x_n, \quad x_{n+1} = x_n + p_{n+1}, \quad (\mathrm{mod}\ 2\pi) \tag{367} $$

which can be derived from

$$ \frac{d^2x}{dt^2} = -K \sin(x) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - (n+\varepsilon) \right), \quad \varepsilon \to 0+ \tag{368} $$

A fractionalization by applying the Riemann-Liouville derivative leads to [186, 189]

$$ {}^{RL}D_t^{\alpha} x = -K \sin(x) \sum_{n=0}^{\infty} \delta\left(\frac{t}{T} - (n+\varepsilon) \right), \quad 1 < \alpha \le 2, \quad \varepsilon \to 0+ \tag{369} $$

with initial conditions ${}^{RL}D_t^{\alpha-1} x(0+) = p_1$ and ${}^{RL}D_t^{\alpha-2} x(0+) = b$.

Otherwise, with the application of the Caputo derivative, we get [189]

$$^{C}D_t^{\alpha}x = -K\sin(x)\sum_{n=0}^{\infty}\delta\left(\frac{t}{T}-(n+\varepsilon)\right), \quad 1<\alpha\leq 2, \quad \varepsilon\to 0+ \tag{370}$$

with initial conditions $p(0) = {}^{C}D_t^{\alpha-1}x(0) = \frac{dx(0)}{dt} = p_0$ and ${}^{RL}D_t^{\alpha-2}x(0+) = b$, where $x(0) = x_0$.
The integration of Eq.(367) yields [189]

$$p_{n+1} = p_n - K\sin(x_n), \quad x_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}p_{k+1}V_{\alpha}^{1}(n-k-1), \quad (\bmod 2\pi) \tag{371}$$

with $V_{\alpha}^{K} = m^{\alpha-k} - (m-1)^{\alpha-k}$ and $p(t) = {}^{RL}D_t^{\alpha-1}x(t)$; $T=1$, and $1<\alpha\leq 2$.
Similarly, the integration of Eq.(370) , with $p = dx/dt$, $T=1$ and $1<\alpha\leq 2$, results in

$$p_{n+1} = p_n - K\frac{1}{\Gamma(\alpha-1)}\left[\sum_{k=0}^{n-1}V_{\alpha}^{2}(n-k+1)\sin(x_k) + \sin(x_n)\right], \quad (\bmod 2\pi) \tag{372}$$

$$x_{n+1} = x_n - K\frac{1}{\Gamma(\alpha)}\left[\sum_{k=0}^{n}V_{\alpha}^{1}(n-k+1)\sin(x_k)\right], \quad (\bmod 2\pi) \tag{373}$$

An alternative form of the map (372)-(373) is

$$x_{n+1} = \sum_{k=0}^{n-1}\frac{c_k}{k!}h^k(n+1)^k - \frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{n}G_k(x_k)(n-k+1)^{\alpha-1} \tag{374}$$

which can be obtained from a version of Eq.(370) related to a periodically kicked system (T is the period) [177, 185, 190]

$$^{C}D_t^{\alpha}x(t) + G_k[x(t-\Delta t)]\sum_{n=-\infty}^{\infty}\delta\left(\frac{t}{T}-(n+\varepsilon)\right) = 0 \tag{375}$$

where $\varepsilon > \Delta > 0$, $\varepsilon\to 0$, $\alpha\in\mathbb{R}$, $0\leq n-1<\alpha<n$. It is easy to obtained from Eq.(374) with $G_k(x) = x - Kx(q-x)$ [189].

10.2.5. Stable fixed point

The standard map as well as the fractionalized versions with ${}^{C}D_t^{\alpha}x$ and ${}^{RL}D_t^{\alpha}x$ have the same fixed point at (0, 0) [185,189]. For the standard map, this point is stable for $K < K_{critical} < 4$. With the fractionalized map involving ${}^{RL}D_t^{\alpha}x$the evolution trajectories near the fixed point are [189]

$$\delta p_{n+1} = \delta p_n - K\delta x_n, \quad \delta x_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\delta p_{n+1}V_{\alpha}(n-k+1) \tag{376}$$

With ${}^{C}D_t^{\alpha}x$ the evolution trajectories near the fixed point are [189]

$$\begin{aligned}\delta p_{n+1} &= \delta p_n - K\frac{1}{\Gamma(\alpha-1)}\left[\sum_{k=0}^{n-1}V_{\alpha}^{2}(n-k+1)\,\delta x_k + \delta x_n\right]\\ \delta x_{n+1} &= \delta x_n + \delta p_0 - K\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n-1}V_{\alpha}^{1}(n-k+1)\,\delta x_k\end{aligned} \tag{377}$$

More detailed analysis and numerical simulations are available in [189].

Remark 24 (On the fractionalization of the maps)**.** *We see that the fractionalization of the map is made directly by the replacement of the integer-order derivatives by fractional ones. For example, the case with Eq.(367) was developed through (see the explanation in [186]) a solution to the Cauchy problem [192]*

$$^{RL}D_t^{\alpha} = G[t, x(t)] \tag{378}$$

with initial conditions ${}^{RL}D_t^{\alpha-k}(x)(0+) = c_k$, $k = 1, 2, \ldots n-1$.
Precisely, ${}^{RL}D_t^{\alpha-k}(x)(0+) = \lim_{t\to 0+}{}^{RL}D_t^{\alpha-k}x(t)$ *and* ${}^{RL}D_t^{\alpha-n}(x)(0+) = \lim_{t\to 0+}{}^{RL}I_t^{n-\alpha}x(t)$*, that is the limit taken at almost all points of the right-sided vicinity* $(0, 0+\varepsilon)$, $\varepsilon > 0$ *[186]. The problem (378)*

can be reduced to a nonlinear Volterra integral equation of the second kind [186, 192]

$$x(t) = \sum_{k=1}^{n} \frac{c_k}{\Gamma(\alpha - k + 1)} t^{\alpha - k} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{G[s, x(s)]}{(t - s)^{1-\alpha}} ds, \quad t > 0 \tag{379}$$

For $\alpha = 2$, Eq. (379) reduces to Eq.(350) (see Remark 22).

For understanding the background of all these results, we recommend a thorough reading of [186, 192] as well as [185, 187–189]. Moreover, we discuss the power-law memory constructions of logistic maps in the next Section 10.2.6

10.2.6. The Power-law memory kernel in the fractional logistic map construction

According to Edelman [191], the logistic map is a very simple discrete non-linear model of dynamical evolution accounting for the whole history of its development. Now, we focus our attention on the problem concerning the memory kernel function used in the basic non-local differential equation from which the corresponding discrete map can be constructed. Precisely, we address the power-law memory, as it was used in all preceding points of this section, and the main idea is to justify its origin in the fractional logistic maps.

Edelman demonstrates that the direct way to construct a map with a power-law memory is through a convolution, that is, (in the original notations) [191]

$$x_n = \sum_{k=0}^{n-1} (n - k)^{\alpha - 1} G_K(x_k, h) \tag{380}$$

where K is a parameter, while h is the time step between the time instant t_n and t_{n+1}.

The more general construction is [191] (recall Eq.(379))

$$x_n = \sum_{k=1}^{[\alpha]-1} \frac{c_k}{\Gamma(\alpha - k + 1)} (nh)^{\alpha - k} + \sum_{k=0}^{n-1} (n - k)^{\alpha - 1} G_K(x_k, h), \quad \alpha \in \mathbb{R} \tag{381}$$

Note: Remember the elegant appearance of the convolution in the Zwanzig model (Section 5.3.3 and the explanations in Remark 7).

Assuming $G_K(x_k, h) = \frac{1}{\Gamma(\alpha)} h^\alpha G_K(x)$, where $G_K(x)$ is continuous, as well as with $dx/dt = x(t)$, $x_k = x(t_k)$, $t_k = a + kh$, $nh = t - a$, where $0 \leq k \leq n$, when $h \to 0$, we get an equivalent Volterra equation of the second kind [191]

$$x_n = \sum_{k=1}^{[\alpha]-1} \frac{c_k}{\Gamma(\alpha - k + 1)} (t - a)^{\alpha - k} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{G_K(s, x(s))}{(t - s)^{1-\alpha}} ds \tag{382}$$

Also, Eq.(382) is eqauivalento the fractional differential equation [191]

$${}^{RL}D_t^{\alpha-k}(x)(a+) = {}^{GL}D_t^{\alpha-k}(x)(a+) = G_K(t, x(t)), 0 < \alpha \tag{383}$$

with initial conditions ${}^{RL}D_t^{\alpha-k}(x)(a+) = {}^{GL}D_t^{\alpha-k}(x)(a+) = c_k$, $k = 1, 2, \ldots [\alpha]$, valid with the Riemann-Liouville ${}^{RL}D_t^{\alpha-k}(x)$ and Grunvald-Letnikov ${}^{GL}D_t^{\alpha-k}(x)$ derivatives.

11. Final remarks

This systematic review went through many points pertaining to problems involving logistic models and sigmoid functions. Looking at many available texts on logistic models with various applications, we saw that such a compilation of classical expositions of dynamic logistic-type models and non-local formulations has never been created before. Indeed, some topics, such as logistic chaotic maps, fractional Riccati equations related to processes of control, or Bass diffusion models, were not considered, but from a personal perspective, this would be a task for a project bigger than a systematic review. The point on fractional operators involving the sigmoid function was hard to be created due to insufficient published information, but now we hope the collected information will motivate somebody to work on similar problems.

We were interested and highly motivated to demonstrate the links from classical local models to the non-local ones based on the Volterra equations, where the problem of the choice of the memory kernel was not the primary issue. However, in the modern era, where fractional calculus is widely spread in mathematical modeling, the concept of how logistic models could be formulated in terms of fractional operators was highly necessary, and we took a step ahead to explain the basic rules. To this point, we believe that the explanation of how this could be done would be fruitful and will allow a better understanding of how fractionalization of logistic models should be done and, therefore, permit discovering the origins of discrepancies between the solutions developed; we hope the present analys will usefull.

Finally, I would like to recommend further readings on many aspects of dynamic growth models, covering both mathematical aspects and philosophical issues, two systematic studies driven by Elliot Montroll [196, 197] that might enlarge the perspective on logistic modeling despite the fact that are about 50 years old, but still young as expressions and ideas.

The motivation to create a well-structured and comprehensive exposition, as we suggested, showing the links between subjects commonly discussed separately but having common ground at the origin, allowed for the creation of this text. I would be pleased if this systematic approach would create new horizons and new motivations for research.

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Conflict of interest

There is no conflict of interest to disclose.

Author contributions

Jordan Hristov is the sole author of this text, including all parts of it, starting from the concept, analysis, data collection, and interpretations, as well as graphical illustrations.

Declaration of using AI tools

The authors declare that they have not used any type of generative artificial intelligence for the writing of this manuscript, nor for the creation of images, graphics, tables, or their corresponding captions.

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