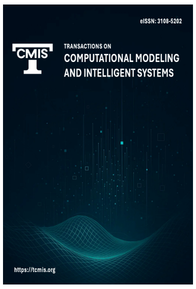

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RESEARCH ARTICLE

Solving fractal fractional differential equations of a function with respect to another function by using the spectral method

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Abstract

The broad applicability of fractional calculus to modeling physical phenomena via fractional differential equations, along with their complexity, has created substantial demand for efficient analytic and semi-analytic techniques for solving them. This paper has derived a numerical approach to solving a specific class of fractal fractional differential equations (FFDEs) involving the Generalized Caputo-type (GC) of a function with respect to another function, or fractal-Ψ-GC, which is shown in this study. The approach is based on an operational matrix (OM) of the fractal fractional derivative of a novel form of orthogonal polynomial. The Ψ-normalized shifted Legendre polynomials (NSLP) and Ψ-shifted Legendre polynomials (SLP) are introduced. The main characteristic of this approach is that it reduces such problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. Examples are provided to portray the efficiency and applicability of this method. Comparison with similar existing approaches is also conducted to demonstrate the accuracy of the proposed approach.

Keywords: Ψ-Generalized Caputo fractional derivative, Ψ-SLP, Ψ-NSLP, OM

1. Introduction

Over the past 30 years, many scientists have shown great interest in fractional calculus, which involves derivatives and integrals of arbitrary order. This growing interest is driven by the significant results of applying fractional calculus to real-world models [1].

One of the key benefits of classical fractional calculus is its extensive range of derivatives and integrals. However, there has always been a need to expand this calculus and discover new derivatives to enhance our understanding of the universe.

Fractional derivatives have become an increasingly important and popular tool in modern research because they extend classical differentiation to non-integer orders, enabling models to capture memory effects and nonlocal behaviors that traditional calculus cannot [2]. This feature makes fractional calculus especially useful in areas such as engineering systems modeling, control theory, and signal processing, where past states influence current dynamics and classical derivatives often fail to describe complex behaviors accurately [3]. Recent works highlight both theoretical advances in fractional derivative definitions and a wide range of applications — from solving nonlinear fractional differential equations to improved modeling in anomalous diffusion and image processing [4]. These developments demonstrate the growing interest and utility of fractional derivatives in both theoretical mathematics and practical scientific applications [2].

In [5], a new class of fractional differential equations (FDEs) was introduced involving the fractional derivative Ψ-GC. The existence, uniqueness, and continuous dependence of the solution to this problem are discussed in [24]. Generalizations of fractional integrals and derivatives of a function with respect to another function, incorporating a weight function, are presented. The OM of integer integration has been applied to various orthogonal polynomials, including Chebyshev polynomials [6], Legendre polynomials [7], and Laguerre and Hermite polynomials [8], among others. Subsequently, many authors extended this approach to fractional cases. For example, see [9–17]. It is important to note that all the cited works focus on solving the Generalized Caputo-type fractional derivatives of a function with respect to another function.

The main contribution of this work is the introduction of a novel Ψ-based operational matrix approach combined with shifted Legendre polynomials. of fractal FDEs of a function with respect to another function.

$$\wp^{\nu,\mu,\Psi}\mathrm{g}(\zeta) = \lambda\mathrm{g}(\zeta) = f(\zeta), \quad 0 < \zeta < 1,$$

under the initial conditions

$$\mathrm{g}^{(\tau)}(0) = d_\tau, \quad \tau = 0, \dots, \Omega - 1$$

where λ is a constant.

In the present paper, we intend to extend the method based on an OM approach for fractal FDEs utilizing a novel class of orthogonal polynomials, the bases of the Ψ-SLP and Ψ-NSLP and derive explicit formulas for the Ψ-fractional differentials of Ψ-SLP and Ψ-NSLP. By projecting the problem onto this polynomial basis, we transform it into an algebraic equation that is easily solvable. Provide a rigorous proof of the convergence of the proposed method. Additionally, numerical experiments demonstrate the procedure's convergence by comparing the exact solution with numerical estimates.

This method differs from existing techniques in the literature by offering enhanced numerical stability and reduced computational cost, making it more effective for solving fractional boundary value problems.

The organization of this paper is as follows: In Section 2, we introduce the Ψ-SLP and the Ψ-NSLP and provide some preliminary concepts in fractional calculus. Section 3 generalizes the OM to Ψ-GC derivatives. Section 4 describes the general procedure for applying the OM to Ψ-SLP. Finally, Section 5 presents numerical results to demonstrate the effectiveness of the proposed approach.

2. Basic concepts

This section defines the fractal-Caputo derivative of a function with respect to another function, also known as the fractal-Ψ-Caputo fractional derivative.

In this context, let $\psi : [a, b] \to \mathbb{R}$ be a strictly increasing and continuously differentiable function such that $\psi'(\zeta) \neq 0$ for all $\zeta \in [a, b]$. These conditions ensure the invertibility of ψ and the well-defined nature of the fractional operator [5]. Furthermore, the fractional operator is defined on an appropriate function space, typically $C^n[a, b]$, to guarantee the existence and uniqueness of the solution [5].

Definition 1. *Let $\nu > 0$ and $\Psi \in C^\tau[a, b]$ be a function such that Ψ is increasing and $\Psi'(\zeta) \neq 0$, for all $\zeta \in [a, b]$ [5].*
If $\zeta \in C^\tau[a, b]$, then the Ψ-Caputo fractional derivative of g can be represented by the expression

$$^{C}\wp^{\nu,\Psi}\mathrm{g}(\zeta) = \frac{1}{\Gamma(\tau - \nu)} \int_0^\zeta \Psi'(s)(\Psi(\zeta) - \Psi(s))^{\tau-\nu-1}\mathrm{g}_\Psi^{(\tau)}(s)ds \tag{1}$$

Next, a fractional derivative of a power function Ψ -Caputo is given. Let $\tau \in \mathbb{R}$ with $\delta > 1$.

$$g(\zeta) = (\Psi(\zeta) - \Psi(a))^{\tau-1}$$

is given by the formula

$$^{C}\wp^{\nu,\Psi} g(\zeta) = \frac{\Gamma(\tau)}{\Gamma(\tau-\nu)}(\Psi(\zeta) - \Psi(a))^{\tau-\nu-1} \tag{2}$$

Also, for Ψ-Caputo, the fractal fractional derivative is given by

$$^{FFC}\wp^{\nu,\mu,\Psi} g(\zeta) = \frac{1}{\Gamma(\tau-\nu)} \int_0^{\zeta} \frac{dg(s)}{ds^{\mu}}(\Psi(\zeta) - \Psi(s))^{-\nu} ds. \tag{3}$$

μ represents the fractal order, the integral has a power law kernel, and,

$$\begin{aligned}\frac{dg(s)}{ds^{\mu}} &= \lim_{\zeta \to s} \frac{g(\zeta) - g(s)}{\zeta^{\mu} - s^{\mu}} \\ &= \frac{1}{\mu\zeta^{\mu-1}} \frac{d}{ds} g(s)\end{aligned}$$

Definition 2. *The novel GC-type fractal fractional derivative of order $\nu > 0$ is defined as [11]:*

$$^{FFGC}\wp^{\nu,\varrho,\mu,\Psi} g(\zeta) = \frac{\varrho^{\nu-\tau+1}}{\Gamma(\tau-\nu)} \int_0^{\zeta} \Psi(s)^{\varrho-1}(\Psi(\zeta)^{\varrho} - \Psi(s)^{\varrho})^{\tau-\nu-1}(\Psi(s)^{1-\varrho}\frac{d}{d\Psi(s)})^{\tau} g_{\Psi}(s) ds, \quad \zeta > 0 \tag{4}$$

where $\varrho > 0$, $\tau - 1 < \nu < \tau$, $\tau = \lceil \nu \rceil$, and $g(\varsigma) \in C^{\tau}[\nu, \varrho]$.

Furthermore, the following is included in the new GC fractional derivative [11]:

$^{FFGC}\wp^{\nu,\varrho,\mu} C = 0$, and C is a constant. Moreover, if $\tau - 1 < \nu < \tau$, $\quad \rho > \tau - 1$ and $k \notin \mathbb{N}$

$$^{FFGC}\wp_{\sigma+}^{\nu,\varrho,\mu,\Psi} (\Psi(\zeta)^{\varrho} - \Psi(a)^{\varrho})^{\tau} = \begin{cases} \varrho^{\nu} \frac{\Gamma(\tau+1)}{\Gamma(\tau-\nu+1)}(\Psi(\zeta)^{\varrho} - \Psi(a)^{\varrho})^{\tau-\nu}, & \tau \in \mathbb{N}_0 \quad and \quad \tau \geq \lceil \nu \rceil \quad or \\ & \tau \in \mathbb{N} \quad and \quad \tau > \lfloor \nu \rfloor \\ 0 \quad , & \tau \in \mathbb{N}_0 \quad and \quad \tau < \lceil \nu \rceil \end{cases} \tag{5}$$

2.1. Ψ-SLP

The following recurrence formulae may be used to locate Legendre polynomials, which are well-known polynomials defined on the interval $[-1, 1]$:

$$L_{\iota+1}(z) = \frac{2\iota+1}{\iota+1} z L_{\iota}(z) - \frac{\iota}{\iota+1} L_{\iota-1}(z), \qquad \iota = 1, 2, \ldots,$$

in [15] which $L_0(z) = 1$ and $L_1(z) = z$. To apply these polynomials to the interval $\zeta \in [0, 1]$, we define the so-called SLP by introducing the change in variable $z = 2\zeta - 1$. Let SLP $L_{\iota}(2\zeta - 1)$ be denoted by $\hbar_{\iota}(\zeta)$. Then $\hbar_{\iota}(\zeta)$ can be obtained as follows:

$$\hbar_{\iota+1}(\zeta) = \frac{(2\iota+1)(2\zeta-1)}{(\iota+1)} \hbar_{\iota}(\zeta) - \frac{\iota}{\iota+1} \hbar_{\iota-1}(\zeta), \qquad \iota = 1, 2, \ldots,$$

where $\hbar_0(\zeta) = 1$ and $\hbar_1(\zeta) = 2\zeta - 1$. The analytic form of the SLP $\hbar_{\iota}(\zeta)$ of degree ι given by

$$\hbar_{\iota}(\zeta) = \sum_{\rho=0}^{\iota} (-1)^{\iota+\rho} \frac{(\iota+\rho)!}{(\iota-\rho)!} \frac{\zeta^{\rho}}{(\rho!)^2}.$$

We introduce Ψ-SLP as follows. The Ψ-SLP $\hbar_{\iota,\Psi}(\zeta)$ are given by

$$\hbar_{\iota,\Psi}(\zeta) = \sum_{\rho=0}^{\iota} (-1)^{\iota+\rho} \frac{(\iota+\rho)!}{(\iota-\rho)!} \frac{(\Psi(\zeta))^{\rho}}{(\rho!)^2}. \tag{6}$$

A function $g(\zeta)$, by use square integrable $[0,1]$, can expressed in terms of Ψ-SLP

$$g(\zeta)=\sum_{q=0}^{\infty} c_q \hbar_{q,\Psi}(\zeta),$$

where the coefficients c_q are provided by

$$c_q=(2q+1)\int_0^1 g(\zeta)\hbar_{q,\Psi}(\zeta)dt, \qquad q=1,2,\ldots.$$

In practice, only the first $(\Omega+1)$-terms SLP are considered. Then we have

$$g(\zeta)=\sum_{q=0}^{\Omega} c_q \hbar_{q,\Psi}(\zeta)=C^T\phi_\Psi(\zeta),$$

When C, the shifted Legendre coefficient vector, and $\phi_\Psi(\zeta)$, the shifted Legendre vector, are each given by

$$\begin{aligned} C^T &= [c_0, c_1, \ldots, c_\Omega], \\ \phi_\Psi(\zeta) &= [\hbar_{0,\Psi}(\zeta), \hbar_{1,\Psi}(\zeta), \ldots, \hbar_{\Omega,\Psi}(\zeta)]^T. \end{aligned} \tag{7}$$

The derivative of the vector $\phi_\Psi(\zeta)$ may be represented as

$$\frac{d\phi_\Psi(\zeta)}{d\zeta}=\wp^{(1)}\phi_\Psi(\zeta),$$

where $\wp^{(1)}$ is the $(\Omega+1)\times(\Omega+1)$ OM of derivative given by

$$\wp^{(1)}=(d_{\iota q})=\begin{cases} 2(2q+1), \quad for \quad q=\iota-\rho, \begin{cases} \rho=1,3,\ldots,\Omega \quad if \quad \Omega \quad odd \\ \rho=1,3,\ldots,\Omega-1 \quad if \quad \Omega \quad even \end{cases} \\ 0, \qquad otherwise. \end{cases}$$

It is clear that

$$\frac{d^\tau\phi_\Psi(\zeta)}{d\zeta^\tau}=(\wp^{(1)})^\tau\phi_\Psi(\zeta),$$

where $\tau\in N$ and the superscript, in $\wp^{(1)}$ powers of a matrix. Thus

$$\wp^{(\tau)}=(\wp^{(1)})^\tau, \quad \tau=1,2,\ldots.$$

2.2. Ψ-NSLP

The analytic form of the NSLP $N_\iota(\zeta)$ of degree ι defined on $[0,1]$, is given by:

$$N_\iota(\zeta)=\sqrt{2\iota+1}\sum_{\rho=0}^{\iota}(-1)^{\iota-\rho}\frac{(\iota+\rho)!}{(\iota-\rho)!}\frac{\zeta^\rho}{(\rho!)^2}.$$

We introduce Ψ-NSLP as follows.

$$N_\iota(\zeta)=\sqrt{2\iota+1}\sum_{\rho=0}^{\iota}(-1)^{\iota-\rho}\frac{(\iota+\rho)!}{(\iota-\rho)!}\frac{(\Psi(\zeta))^\rho}{(\rho!)^2}. \tag{8}$$

In terms of SLP, a function $g(\zeta)$ that is square integrable in $[0,1]$ may be written as

$$g(\zeta)=\sum_{q=0}^{\infty} c_q \hbar_{q,\Psi}(\zeta),$$

where the coefficients c_q are defined as follows

$$c_q=\sqrt{2q+1}\int_0^1 g(\zeta)\hbar_{q,\Psi}(\zeta)d\zeta, \qquad q=1,2,\ldots.$$

Only the first $(\Omega+1)$-terms of SLP are considered in the solution. Then we have

$$g(\zeta)=\sum_{q=0}^{\Omega} c_q \hbar_{q,\Psi}(\zeta)=C^T\varphi_\Psi(\zeta),$$

where C is the shifted Legendre coefficient vector and $\phi_\Psi(\zeta)$ is the shifted Legendre vector are given by

$$C^T = [c_0, c_1, \dots, c_\Omega],$$
$$\varphi_\Psi(\zeta) = [N_{0,\Psi}(\zeta), N_{1,\Psi}(\zeta), \dots, N_{\Omega,\Psi}(\zeta)]^T. \tag{9}$$

The derivative of the vector $\varphi_\Psi(\zeta)$ can be written as

$$\frac{d\varphi_\Psi(\zeta)}{d\zeta} = \wp^{(1)}\varphi_\Psi(\zeta),$$

where $\wp^{(1)}$ is the $(\Omega+1)\times(\Omega+1)$ OM of derivative given by

$$\wp^{(1)} = (d_{\iota j}) = \begin{cases} 2\sqrt{2\iota+1}\sqrt{2q+1}, & for \quad q=\iota-\rho, \begin{cases} \rho = 1,3,\dots,\Omega & if \quad \Omega \quad odd \\ \rho = 1,3,\dots,\Omega-1 & if \quad \Omega \quad even \end{cases} \\ 0, & otherwise. \end{cases}$$

It is clear that

$$\frac{d^\tau \varphi_\Psi(\zeta)}{d\zeta^\tau} = (\wp^{(1)})^\tau \varphi_\Psi(\zeta),$$

where $\tau \in N$ and the superscript, in $\wp^{(1)}$, denotes matrix powers. Thus

$$\wp^{(\tau)} = (\wp^{(1)})^\tau, \quad \tau = 1,2,\dots.$$

3. Generalized OM to Ψ-GC

In this section, we will generalise the OM using Ψ-SLP and Ψ-NSLP of OM to solve Ψ-GC fractal FDEs.

3.1. Ψ-SLP of OM

The main goal of this section is to generalize the Ψ-SLP for Ψ-GC.

Theorem 1 (22)**.** *Let $\phi_\Psi(\zeta)$ be shifted Legendre vector defined in Eq.(4) and also suppose $\nu > 0$ then*

$$\wp^{\nu,\varrho,\mu,\Psi}\phi_\Psi(\zeta) \simeq \wp^{(\nu,\varrho,\mu,\Psi)}\phi_\Psi(\zeta) \tag{10}$$

where $\wp^{(\nu,\varrho,\mu,\Psi)}$ is the $(\Omega+1)\times(\Omega+1)$ OM of fractional derivative in the Ψ-GC sense and is defined as follows:

$$P^\nu = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\rho=\lceil\nu\rceil}^{\lceil\nu\rceil}\vartheta_{\lceil\nu\rceil,0,\rho} & \sum_{\rho=\lceil\nu\rceil}^{\lceil\nu\rceil}\vartheta_{\lceil\nu\rceil,1,\rho} & \dots & \sum_{\rho=\lceil\nu\rceil}^{\lceil\nu\rceil}\vartheta_{\lceil\nu\rceil,\Omega,\rho} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{\rho=\lceil\nu\rceil}^{\iota}\vartheta_{\iota,0,\rho} & \sum_{\rho=\lceil\nu\rceil}^{\iota}\vartheta_{\iota,1,\rho} & \dots & \sum_{\rho=\lceil\nu\rceil}^{\iota}\vartheta_{\iota,\Omega,\rho} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{\rho=\lceil\nu\rceil}^{\Omega}\vartheta_{\Omega,0,\rho} & \sum_{\rho=\lceil\nu\rceil}^{\Omega}\vartheta_{\Omega,1,\rho} & \dots & \sum_{\rho=\lceil\nu\rceil}^{\Omega}\vartheta_{\Omega,\Omega,\rho} \end{pmatrix} \tag{11}$$

where $\vartheta_{\iota,q,\rho}$ is given by

$$\vartheta_{\iota,q,\rho} = (2q+1)\sum_{\ell=0}^{q}\frac{(-1)^{\iota+q+\rho+\ell}(\iota+\rho)!(\ell+q)!}{(\iota-\rho)!\rho!\Gamma(\rho-\nu+1)(q-\ell)!(\ell!)^2}\int_0^1(\Psi(\zeta))^{\rho+\ell-\nu}d\zeta \tag{12}$$

Note that $\wp^{(\nu,\varrho,\mu,\Psi)}$, the first $\lceil\nu\rceil$ rows, are all zero.

Proof. Using Eqs. (2) and (4), we have

$$\wp^{\nu,\varrho,\Psi}\hbar_{\iota,\Psi}(\zeta)=\sum_{\rho=0}^{\iota}\frac{(-1)^{\iota+\rho}(\iota+\rho)!}{(\iota-\rho)!(\rho!)^2}\wp^{\nu,\varrho,\Psi}(\Psi(\zeta))^{\rho}=\sum_{\rho=\lceil\nu\rceil}^{\iota}\frac{(-1)^{\iota+\rho}(\iota+\rho)!}{(\iota-\rho)!(\rho!)\Gamma(\rho-\nu+1)}\wp^{\nu,\varrho,\Psi}(\Psi(\zeta))^{\rho-\nu},$$
$$\iota=\lceil\nu\rceil,\dots,\Omega.$$

To approximate $(\Psi(\zeta))^{\rho-\nu}\simeq\sum_{q=0}^{\Omega}b_{\rho,q,\Psi}\hbar_{q,\Psi}(\zeta),$

$$b_{\rho,q,\Psi}=(2q+1)\int_0^1(\Psi(\zeta))^{\rho-\nu}\hbar_{q,\Psi}(\zeta)d\zeta=(2q+1)\sum_{\ell=0}^{q}\frac{(-1)^{q+\ell}(q+\ell)!}{(q-\ell)!(\ell!)^2}\int_0^1(\Psi(\zeta))^{\rho+\ell-\nu}d\zeta$$

$$\wp^{\nu,\varrho}\hbar_{\iota,\Psi}(\zeta)\simeq\sum_{\rho=\lceil\nu\rceil}^{\iota}\sum_{q=0}^{\Omega}\frac{(-1)^{\iota+\rho}(\iota+\rho)!}{(\iota-\rho)!(\rho!)\Gamma(\rho-\nu+1)}b_{\rho,q,\Psi}\hbar_{q,\Psi}(\zeta)$$
$$=\sum_{q=0}^{\Omega}\left(\sum_{\rho=\lceil\nu\rceil}^{\iota}\vartheta_{\iota,q,\rho}\right)\hbar_{q,\Psi}(\zeta),\quad\iota=\lceil\nu\rceil,\dots,\Omega,$$

where $\vartheta_{\iota,q,\rho}$ is given in Eq.(8) □

3.2. Ψ-NSLP of OM

The main goal of this section is to generalize Ψ -NSLP for Ψ-GC.

Theorem 2 (22)**.** *Let $\varphi_\Psi(\zeta)$ be Normalized shifted Legendre vector defined in Eq.(6) and also suppose $\nu>0$ then*

$$\wp^{\nu,\varrho,\mu,\Psi}\varphi_\Psi(\zeta)\simeq\wp^{(\nu,\varrho,\mu,\Psi)}\varphi_\Psi(\zeta)\tag{13}$$

where $\wp^{(\nu,\varrho,\mu,\Psi)}$ is the $(\Omega+1)\times(\Omega+1)$ OM of fractional derivative in the Ψ-GC sense and is defined as Eq.(9):
where $\vartheta_{\iota,q,\rho}$ is given by

$$\vartheta_{\iota,q,\rho}=\sqrt{2\iota+1}\sqrt{2q+1}\sum_{\ell=0}^{q}\frac{(-1)^{\iota+q+\rho+\ell}(\iota+\rho)!(\ell+q)!}{(\iota-\rho)!\rho!\Gamma(\rho-\nu+1)(q-\ell)!(\ell!)^2}\int_0^1(\Psi(\zeta))^{\rho+\ell-\nu}d\zeta\tag{14}$$

Note that P^ν, the first $\lceil\nu\rceil$ rows, are all zero.

Proof. Using Eqs. (2) and (4), we have

$$\wp^{\nu,\varrho,\Psi}\hbar_{\iota,\Psi}(\zeta)=\sqrt{2\iota+1}\sum_{\rho=0}^{\iota}\frac{(-1)^{\iota+\rho}(\iota+\rho)!}{(\iota-\rho)!(\rho!)^2}\wp^{\nu,\varrho,\Psi}(\Psi(\zeta))^{\rho}$$
$$=\sqrt{2\iota+1}\sum_{\rho=\lceil\nu\rceil}^{\iota}\frac{(-1)^{\iota+\rho}(\iota+\rho)!}{(\iota-\rho)!(\rho!)\Gamma(\rho-\nu+1)}\wp^{\nu,\Psi}(\Psi(\zeta))^{\rho-\nu},\quad\iota=\lceil\nu\rceil,\dots,\Omega.$$

To approximate $(\Psi(\zeta))^{\rho-\nu}\simeq\sum_{q=0}^{\Omega}b_{\rho,q,\Psi}\hbar_{q,\Psi}(\zeta),$

$$b_{\rho,q,\Psi}=\sqrt{2q+1}\int_0^1(\Psi(\zeta))^{\rho-\nu}\hbar_{q,\Psi}(\zeta)d\zeta=\sqrt{2q+1}\sum_{\ell=0}^{q}\frac{(-1)^{q+\ell}(q+\ell)!}{(q-\ell)!(\ell!)^2}\int_0^1(\Psi(\zeta))^{\rho+\ell-\nu}d\zeta$$

$$\wp^{\nu,\varrho}\hbar_{\iota,\Psi}(\zeta)\simeq\sqrt{2\iota+1}\sqrt{2q+1}\sum_{\rho=\lceil\nu\rceil}^{\iota}\sum_{q=0}^{\Omega}\frac{(-1)^{\iota+\rho}(\iota+\rho)!}{(\iota-\rho)!(\rho!)\Gamma(\rho-\nu+1)}b_{\rho,q,\Psi}\hbar_{q,\Psi}(\zeta)$$
$$=\sum_{q=0}^{\Omega}\left(\sum_{\rho=\lceil\nu\rceil}^{\iota}\vartheta_{\iota,q,\rho}\right)\hbar_{q,\Psi}(\zeta),\quad\iota=\lceil\nu\rceil,\dots,\Omega,$$

where $\vartheta_{\iota,q,\rho}$ is given in Eq.(12) □

4. The general procedure of Ψ-SLP of OM

First, the unknown function $\mathrm{g}(\zeta)$ is approximated by SLP,

$$\mathrm{g}(\zeta) = C^T \phi_\Psi(\zeta)$$

the fractal Ψ-Caputo operator of equation is approximated using the Ψ-SLP of OM as

$$^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi}\mathrm{g}(\zeta) = C^T P^\nu \phi_\Psi(\zeta).$$

The remaining single-variable functions are approximated in Ψ-SLP accordingly:

$$q(\zeta) = Q^T \phi_\Psi(\zeta)$$
$$Q = [q_0, \ldots, q_\Omega].$$

By substituting each term with its Ψ-SLP expansion, the following is obtained:
To do this, we use the following linear equation: the residual $R_\Omega(\zeta)$ can be written as:

$$R_\Omega(\zeta) \simeq \left(C^T \sum_{q=1}^{\rho} a_q P^\nu - a_{\rho+1} C^T - a_{\rho+2} G^T \right) \phi_\Psi(\zeta).$$

We get $\Omega - \tau$ linear equations by using

$$C^T \sum_{q=1}^{\rho} a_q P^\nu - a_{\rho+1} C^T - a_{\rho+2} G^T = 0.$$

This produces a linear system of Ω algebraic equations.
For non-linear, consider the following

$$^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) \simeq F(\zeta, \mathrm{g}(\zeta), \ldots),$$

It should be remembered that F may generally be nonlinear.

$$C^T P^\nu \phi_\Psi(\zeta) \simeq F(\zeta, C^T \phi_\Psi(\zeta), \ldots).$$

We initially collocate at $(\Omega - \tau)$ points. For suitable collocation points, we use the first roots of $(\Omega - \tau)$ Ψ-SLP of $p_{\Omega+1}(\zeta)$. This produces a system non-linear of Ω algebraic equations.
The initial condition given is also approximated in Ψ-SLP:

$$\mathrm{g}^{(\tau)}(0) = d_\tau, \quad \tau = 0, \ldots, \Omega - 1$$
$$C^T \phi_\Psi(0) \wp^\tau = d_\tau.$$

Selecting $\Omega - \tau$ equations combined with initial conditions, one has a system of Ω linear algebraic equations to be solved for C using any suitable numerical methods, such as the Gaussian-Elimination method, and one has a system of Ω non-linear algebraic equations to be solved for C using Newton's iterative method.
The approximate solution can be computed by substituting the coefficient C:

$$\mathrm{g}(\zeta) = C^T \phi_\Psi(\zeta). \tag{15}$$

4.1. Error bound

The error bound for the FDEs operational matrix is presented in this section. The following theorem [22] is provided for this purpose.

Theorem 3. *The error* $|\triangle_\Omega| = |\, ^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) - \, ^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}_\Omega(\zeta)|$ *in approximating* $^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)$ *having the OM of the fractional derivative is bounded as expressed below:*

$$|\triangle_\Omega| \leq \sum_{\iota=\Omega+1}^{\infty} |c_\iota| \sum_{q=1}^{\Omega} |\wp_{\iota,q}| \sum_{\rho=1}^{q} \left| \frac{(-1)^{q+\rho}(q+\rho)!}{(q-\rho)!(\rho!)^2} \right|,$$

in which g_Ω *resembles the h function approximation depending on the SLPs,* c_ι. *Here,* $\iota = 1, 2, \ldots, \Omega$ *resemble the coefficients of this approximation in which*

$$\wp_{\iota,q} = \begin{cases} \sum_{\rho=\lceil\nu\rceil}^{\iota} \vartheta_{\iota,q,\rho}, & for \quad \iota = \lceil\nu\rceil, \ldots, \Omega, \quad q = 1, \ldots, \Omega, \\ 0, & for \quad \iota = 1, \ldots, \lceil\nu\rceil, \quad q = 1, \ldots, \Omega. \end{cases}$$

Proof. By using Eq.(13) can get

$$\mathrm{g}(\zeta)=\sum_{\iota=1}^{\infty}c_\iota P_\iota(\zeta),$$

$$^{FFGC}{}_{\wp}{}^{\nu,\varrho,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)=\sum_{\iota=1}^{\infty}c_\iota\sum_{q=1}^{\Omega}D_{\iota,q}P_q.$$

By considering only the first Ω terms of the infinite series given above, the following can now be obtained

$$^{FFGC}{}_{\wp}{}^{\nu,\varrho,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)-\sum_{\iota=1}^{\Omega}c_\iota\sum_{q=1}^{\Omega}\wp_{\iota,q}P_q(t)=\sum_{\iota=\Omega+1}^{\infty}c_\iota\sum_{q=1}^{\Omega}\wp_{\iota,q}P_q(\zeta) \tag{16}$$

Employing Eq.(13) and Eq.(14) can be illustrated to have a matrix form as given below:

$$^{FFGC}{}_{\wp}{}^{\nu,\varrho,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)-C^T\ {}^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\phi(\zeta)=\sum_{\iota=\Omega+1}^{\infty}c_\iota\sum_{q=1}^{\Omega}\wp_{\iota,q}P_q(\zeta).$$

Now, can we get:

$$|\ {}^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)-C^T{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\phi(\zeta)|=|\sum_{\iota=\Omega+1}^{\infty}c_\iota\sum_{q=1}^{\Omega}\wp_{\iota,q}P_q(\zeta)|\leq\sum_{\iota=\Omega+1}^{\infty}|c_\iota|\sum_{q=1}^{\Omega}|\wp_{\iota,q}||P_q(\zeta)|. \tag{17}$$

The following is an upper bound for SLPs:

$$|P_q(\zeta)|=\left|\sum_{\rho=1}^{q}\frac{(-1)^{q+\rho}(q+\rho)!}{(q-\rho)!(\rho!)^2}\zeta^\rho\right|\leq\sum_{\rho=1}^{q}\left|\frac{(-1)^{q+\rho}(q+\rho)!}{(q-\rho)!(\rho!)^2}\right||\zeta^\rho|\leq\sum_{\rho=1}^{q}\left|\frac{(-1)^{q+\rho}(q+\rho)!}{(q-\rho)!(\rho!)^2}\right|. \tag{18}$$

Thus, by substituting Eq.(15) in to Eq.(16) yields:

$$|\ {}^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)-C^T{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\phi(\zeta)|\leq\sum_{\iota=\Omega+1}^{\infty}|c_\iota|\sum_{q=1}^{\Omega}|\wp_{\iota,q}|\sum_{\rho=1}^{q}\left|(-1)^{q+\rho}\frac{(q+\rho)!}{(q-\rho)!(\rho!)^2}\right|.$$

Hence, the obtained result is

$$|\ {}^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)-\ {}^{FFGC}{}_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}_\Omega(\zeta)|\leq\sum_{\iota=\Omega+1}^{\infty}|c_\iota|\sum_{q=1}^{\Omega}|\wp_{\iota,q}|\sum_{\rho=1}^{q}|(-1)^{q+\rho}\frac{(q+\rho)!}{(q-\rho)!(\rho!)^2}|.$$

Thus, the proof is complete. □

5. Numerical examples

This section will provide some solutions to the numerical examples of linear and non-linear fractal fractional-order cases. The absolute error will be used in our computational results to quantify the difference between the exact and approximate solutions. All the numerical programs are coded and run in MATLAB R2020b software.

- **NLOM** Normalized Legendre Operational matrix method derived in this study.
- **LOM** Legendre Operational matrix method derived in this study.
- **PRCO** Predictor-Corrector [13].
- **NS** Numerical schemes for chaotic attractors [14].
- **LWOMM** Legendre wavelet operational matrix method [19].
- **LWPT** Legendre Wavele-Polynomial Transformation [20].
- **PLSM** Polynomial Least Squares Method [21].
- **OMFI** Operational Matrix of Fractional Integration [9].

Example 1. *Let us consider the FDEs*

$$_{\wp}{}^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta)=\Psi(\zeta),\quad 0<\nu<1 \tag{19}$$

under the initial condition $\mathrm{g}(0)=0$ *and exact solution* $\mathrm{g}(\zeta)=\frac{\mu\Gamma(1+\mu)}{\Gamma(1+\nu+\mu)}(\Psi(\zeta))^{\nu+\mu}$. *The case of* $\Psi(\zeta)=\zeta,\quad \Psi(\zeta)=\log(\zeta+1)$ *and* $\Psi(\zeta)=\tan(\frac{\pi\zeta}{4})\quad \zeta\in I=[0,1]$. *The absolute error and* *comparison with two methods in the case $\Omega = 6$ are shown in Tables 1 and 2 with Figures 1 and, 2.*

Table 1. The absolute error of the proposed method in comparison with other numerical methods for $\nu = 0.8$, $\mu = 0.9$, $\varrho = 1$ and $\Psi(\zeta) = \zeta$ in Example 1.

ζ	NLOM	LOM	PRCO	NS
0.1	6.91010e-5	6.91011e-5	4.69490e-4	3.77435e-4
0.2	2.19954e-4	2.19955e-4	4.62190e-4	3.20234e-4
0.3	1.25996e-6	1.26021e-6	4.57334e-4	2.92185e-4
0.4	1.87139e-4	1.87138e-4	4.52445e-4	2.74114e-4
0.5	1.58079e-4	1.58079e-4	4.47815e-4	2.61001e-4
0.6	2.46070e-5	2.46075e-5	4.43518e-4	2.50820e-4
0.7	1.64394e-4	1.64395e-4	4.39549e-4	2.42561e-4
0.8	1.00567e-4	1.00567e-4	4.35879e-4	2.35650e-4
0.9	1.02217e-4	1.02216e-4	4.32475e-4	2.29733e-4

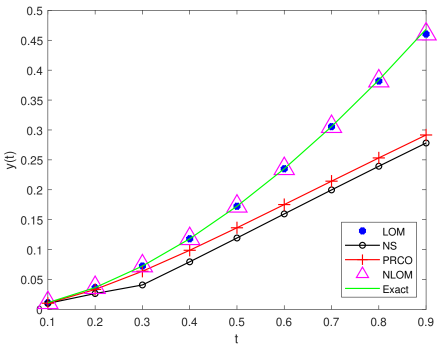

Figure 1. In the case where $\Psi(\zeta) = \log(\zeta + 1)$, the obtained results are in approximate solution with the exact solution and other methods in Example 1, confirming the accuracy of the proposed method.

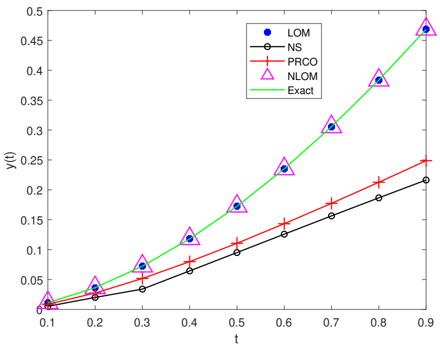

Figure 2. In the case where $\Psi(\zeta) = \tan!\left(\frac{\pi\zeta}{4}\right)$, the proposed method yields an accurate approximate solution compared with the exact solution and other methods in Example 1.

Example 2. *Let us consider the fractional oscillator equation [9].*

$$\wp^{1.5,1.1,1,\Psi(\zeta)}\mathrm{g}(\zeta) + \frac{2}{\Gamma(\frac{3}{2})}\mathrm{g}(\zeta) = \frac{2}{\Gamma(\frac{3}{2})}\sqrt{\Psi(\zeta)}(1 + (\Psi(\zeta))^{\frac{3}{2}})$$

under the initial conditions

$$\mathrm{g}(0) = \mathrm{g}'(0) = 0,$$

Table 2. presents the absolute error of the proposed method for $\alpha = 0.8$, $\beta = 0.9$, $\varrho = 1$ and $\Psi(\zeta) = \zeta$ for Example 1 with different Ω.

ζ	$\Omega = 2$	$\Omega = 4$	$\Omega = 6$
0.1	5.13150e-3	1.23809e-3	6.91011e-5
0.2	5.16693e-3	1.82663e-4	2.19955e-4
0.3	3.17005e-3	8.47520e-4	1.26021e-6
0.4	4.37056e-4	1.30123e-3	1.87138e-4
0.5	2.22335e-3	1.11474e-3	1.58079e-4
0.6	4.23543e-3	4.57323e-4	2.46075e-5
0.7	5.15850e-3	3.66765e-4	1.64395e-4
0.8	4.63916e-3	9.65921e-4	1.00567e-4
0.9	2.38468e-3	8.87901e-4	1.02216e-4

and the exact solution is written as g^*. *It's easy to see that*

$$\mathrm{g}^*(\zeta) = (\Psi(\zeta))^2.$$

In this example, different choices of the function Ψ *are considered in* $\Psi(\zeta) = \zeta$, $\zeta/2(\zeta + 1)$ *and* $\tan(\frac{\pi\zeta}{4})$. *Table 3 with Figures 3 and 4 display the absolute error and its comparison to two other methods for* $\Omega = 6$.

Table 3. The absolute error of the proposed method in comparison with other numerical methods for $\nu = 1.5$, $\mu = 1.1$, $\varrho = 1$ and $\Psi(\zeta) = \zeta$ in Example 2.

ζ	NLOM	LOM	PRCO	NS
0.1	6.13455e-4	6.13455e-4	3.87899e-3	1.81709e-2
0.2	1.88915e-3	1.88915e-3	8.23867e-3	4.75961e-2
0.3	2.79516e-3	2.79516e-3	1.17913e-2	8.36696e-2
0.4	2.10340e-3	2.10340e-3	1.39555e-2	1.25223e-1
0.5	1.54737e-3	1.54737e-3	1.42470e-2	1.71601e-1
0.6	9.58861e-3	9.58861e-3	1.23664e-2	2.22214e-1
0.7	2.34588e-2	2.34588e-2	8.16711e-3	2.76470e-1
0.8	4.45410e-2	4.45410e-2	1.62759e-3	3.33761e-1
0.9	7.40989e-2	7.40989e-2	7.17211e-3	3.93474e-1

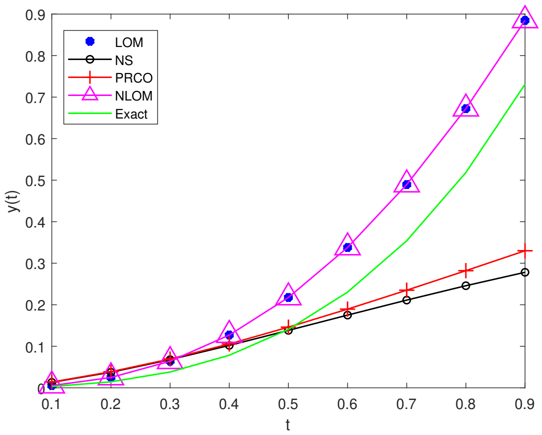

Figure 3. For $\Psi(\zeta) = \frac{\zeta}{2}(\zeta + 1)$, the method provides an accurate approximate solution compared with the exact solution and other methods in Example 2.

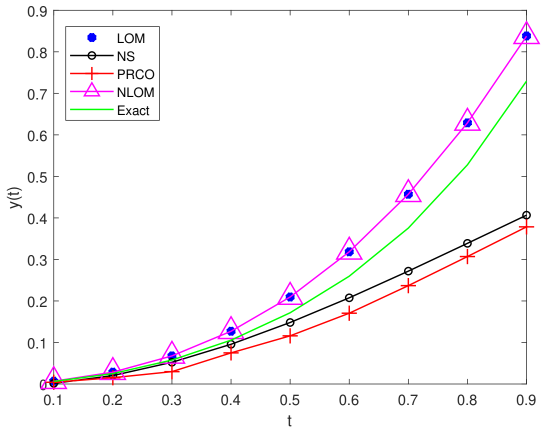

Figure 4. For $\Psi(\zeta) = \tan!\left(\frac{\pi\zeta}{4}\right)$, the method provides an accurate approximate solution compared with the exact solution and other methods in Example 2.

Example 3. *Let the non-linear fractal fractional Riccati the differential equation is given as follows*

$$\wp^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) = 1 - \mathrm{g}^2(\zeta)$$

Here, the exact solution is provided by $\mathrm{g}(\zeta) = \frac{e^{2(\Psi(\zeta))}-1}{e^{2(\Psi(\zeta))}+1}$ *and* $\mathrm{g}(0) = 0$ *initial condition. Here, the approximation of* $\nu = 0.99, \quad \varrho = 1, \quad \mu = 0.98.$ *The absolute error and the comparison with two methods in the case* $\Omega = 6$ *are shown in Tables 4 and Figures 5 and 6.*

Table 4. The absolute error of the proposed method in comparison with other numerical methods for $\nu = 0.99, \quad , \varrho = 1 \ \mu = 0.98$ and $\Psi(\zeta) = \zeta$ in Example 3.

ζ	NLOM	LOM	PRCO	NS
0.1	3.97861e-3	3.97861e-3	5.25980e-3	3.60109e-3
0.2	6.50175e-3	6.50175e-3	7.93716e-3	5.55350e-3
0.3	7.36001e-3	7.36001e-3	8.91243e-3	6.16823e-3
0.4	6.68083e-3	6.68083e-3	8.62695e-3	5.76870e-3
0.5	4.76054e-3	4.76054e-3	7.39663e-3	4.62432e-3
0.6	1.94567e-3	1.94567e-3	5.52752e-3	2.99160e-3
0.7	1.43931e-3	1.43931e-3	3.31087e-3	1.10546e-3
0.8	5.12901e-3	5.12901e-3	1.00177e-3	8.35417e-4
0.9	8.93271e-3	8.93271e-3	1.19884e-3	2.67834e-3

Example 4. *Let the non-linear fractal fractional Riccati the differential equation is given as follows*

$$\wp^{\nu,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) = 2\mathrm{g}(\zeta) + 1 - g^2(\zeta)$$

Here, the exact solution is provided by $\mathrm{g}(\zeta) = 1+\sqrt{2}\tanh(\sqrt{2}(\Psi(\zeta)))+0.5\log\frac{\sqrt{2}-1}{\sqrt{2}+1})$ *and* $\mathrm{g}(0) = 0$ *initial condition. Here, the approximation of* $\nu = 0.99, \quad \varrho = 1, \quad \mu = 0.96.$ *The absolute error and comparison with two methods in the case* $\Omega = 6$ *are shown in Table 5 and Figures 7 and 8.*

Example 5. *Let us consider the fractal-fractional differential equation [23]*

$$\wp^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) = (\Psi(\zeta))^3, \quad 0 < \nu < 1$$

under the initial conditions

$$\mathrm{g}(0) = \mathrm{g}^{'}(0) = 0,$$

and we denote by g^* *the exact solution. It can be easily seen that*

$$\mathrm{g}(\zeta) = \frac{\mu\Gamma(3+\mu)}{\Gamma(3+\nu+\mu)}(\Psi(\zeta))^{\nu+\mu+2}.$$

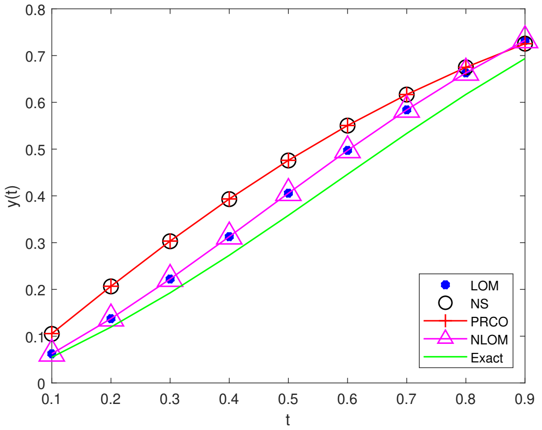

Figure 5. In case, $\Psi(\zeta) = \frac{\zeta}{2}(\zeta+1)$, we obtained a good approximation when compared with the exact solution and different methods for Example 3

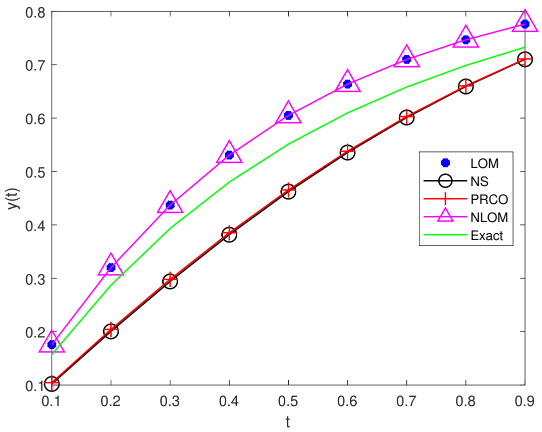

Figure 6. In the case, $\Psi(\zeta) = \log((e-1)\zeta+1)$, we obtained a good approximation when compared with the exact solution and different methods for Example 3.

Table 5. Approximate solutions for $\nu = 0.99, \quad \varrho = 1, \quad \mu = 0.96$ and $\Psi(\zeta) = \zeta$ in Example 4.

ζ	Approximate solutions(NLOM)	Approximate solutions(LOM)	Exact solutions	Approximate solutions(PRCO)	Approximate solutions(NS)
0.1	0.125541	0.122152	0.110295	0.107630	0.103722
0.2	0.270106	0.263596	0.241976	0.242225	0.235552
0.3	0.436858	0.427643	0.395104	0.397136	0.388551
0.4	0.623847	0.612461	0.567812	0.570770	0.560337
0.5	0.825367	0.812433	0.756014	0.759135	0.746817
0.6	1.033320	1.019517	0.953566	0.956053	0.941850
0.7	1.238572	1.224610	1.152948	1.153836	1.137935
0.8	1.432319	1.418902	1.346363	1.344449	1.327378
0.9	1.607445	1.595239	1.526911	1.520887	1.503547

Different choices of the function Ψ are $\Psi(\zeta) = \zeta$, $\zeta/2(\zeta+1)$ and $\tan(\frac{\pi\zeta}{4})$ considered in this example. The absolute error and comparison with two methods in the case $\Omega = 6$ are shown in Table 6 and Figures 9 and 10.

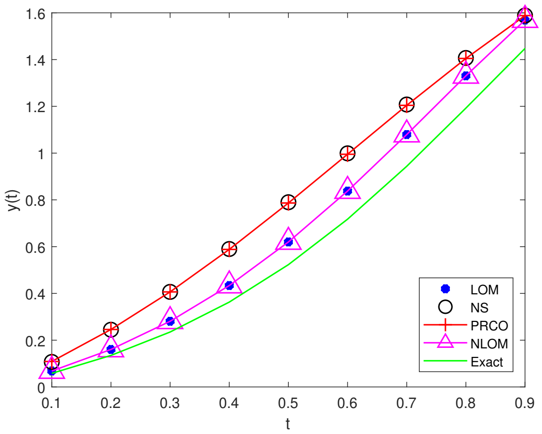

Figure 7. In the case, $\Psi(\zeta) = \frac{\zeta}{2}(\zeta + 1)$, we obtained a good approximation when compared with the exact solution and different methods for Example 4

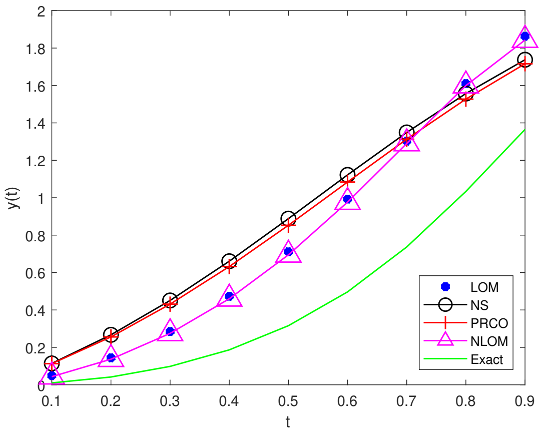

Figure 8. In the case, $\Psi(\zeta) = \zeta^2$, we obtained a good approximation when compared with the exact solution and different methods for Example 4

Table 6. Comparing the absolute error between NLOM and LOM for $\alpha = 0.95$, , $\varrho = 1$ $\beta = 0.95$ and $\Psi(\zeta) = \zeta$ with other methods in Example 5.

ζ	NLOM	LOM	PRCO	NS
0.1	3.43147e-7	3.43287e-7	2.99727e-5	3.26187e-5
0.2	4.40839e-7	4.40987e-7	1.13379e-4	4.86950e-4
0.3	4.79853e-7	4.79697e-7	2.45992e-4	1.00297e-3
0.4	5.54316e-7	5.54151e-7	4.25772e-4	1.77050e-3
0.5	1.85375e-7	1.85548e-7	6.51355e-4	2.77313e-3
0.6	6.08449e-7	6.08629e-7	9.21719e-4	4.00374e-3
0.7	1.30968e-7	1.31157e-7	1.23604e-3	5.45728e-3
0.8	4.90383e-7	4.90187e-7	1.59366e-3	7.12975e-3
0.9	6.08130e-8	6.06096e-8	1.99399e-3	9.01787e-3

Example 6. *The Brusselator system is a well-known model of a chemical reaction. The following is a description of the Brusselator chemical reaction system model with fractional-order [18]:*

$$\wp^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) = -2\mathrm{g}(\zeta) + \mathrm{g}^2(\zeta)\mathrm{g}(\zeta),$$
$$\wp^{\nu,\varrho,\mu,\Psi(\zeta)}\mathrm{g}(\zeta) = \mathrm{g}(\zeta) - \mathrm{g}^2(\zeta)\mathrm{g}(\zeta),$$

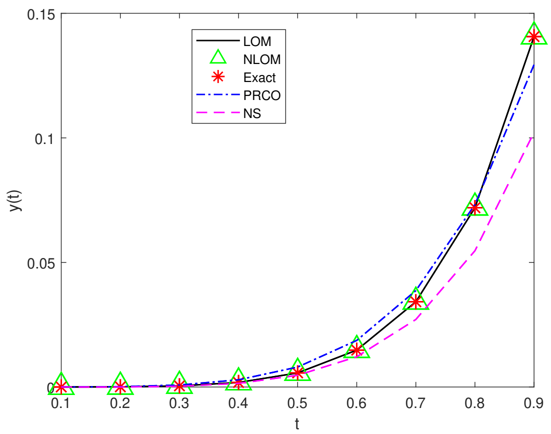

Figure 9. Comparing the approximate solutions of NLOM and LOM with the exact solutions, as well as other methods, for $\Psi(\zeta) = \frac{\zeta}{2}(\zeta + 1)$ in Example 5.

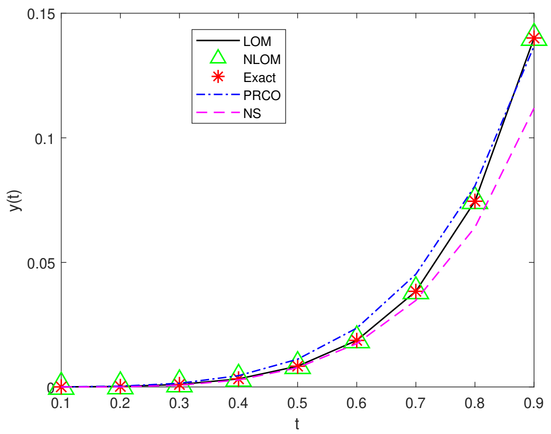

Figure 10. Comparing the approximate solutions of NLOM and LOM with the exact solutions, as well as other methods, for $\Psi(\zeta) = \tan(\frac{\pi\zeta}{4})$ in Example 5.

the initial conditions: $\mathrm{g}(0) = 1$, $\mathrm{g}(0) = 1$.
The approximate solutions of this system when $\nu = 1$ *,* $\varrho = 1$ *and* $\nu = 0.98$ *were presented by Chang and Isah using the LWOMM [19], LWPT [20] and by Bota and Caruntu using the PLSM [21].*
The parameters $\Omega = 2$ *are used. A comparison of our results with the approximate solutions introduced by Chang and Isah using LWOMM [19], LWPT [20] and Bota and Caruntu using PLSM [21] when* $\alpha = 0.98$ *is shown in Figures 11 and 12. Finally, we also present the numerical calculations for* $\mathrm{g}(\zeta)$ *and* $\mathrm{g}(\zeta)$ *when* $\alpha = 0.98$ *and* $\Psi(\zeta) = \zeta^{\delta}$*, where* $\delta = 1$ *in Tables 7 and 8.*

Advantages of the proposed method

The proposed LOM method offers several advantages. It provides highly accurate approximate solutions for fractional differential equations while maintaining computational efficiency. The method is straightforward to implement and demonstrates reliable convergence for different choices of the Ψ-function. Additionally, it allows flexibility in handling various fractional orders and boundary conditions. However, like most spectral-based approaches, the computational effort increases for higher-order approximations or more complex systems. Despite this limitation, the overall performance confirms that the method is robust, efficient, and effective for solving a wide range of fractional differential equations.

Table 7. the approximate solutions of g(ζ) for $\nu = 0.98$, $\varrho = 1$ $\mu = 1$ and $\Psi(\zeta) = \zeta^{\delta}$ in Example 6.

ζ	LOM	LWOMM	LWPT	PLSM
0.1	0.8919967302	0.8942024826	0.8947372	0.8944807482
0.2	0.7912617286	0.7950696916	0.7945136	0.7953304656
0.3	0.6977949953	0.7026016268	0.6989464	0.7026953614
0.4	0.6115965303	0.6167982883	0.6076528	0.6167216448
0.5	0.5326663335	0.5376596761	0.5202500	0.5375555250
0.6	0.4610044049	0.4651857902	0.4363552	0.4653432112
0.7	0.3966107446	0.3993766306	0.3555856	0.4002309126
0.8	0.3394853526	0.3402321973	0.2775584	0.3423648384
0.9	0.2896282288	0.2877524902	0.2018908	0.2918911978

Table 8. the approximate solutions of g(ζ) for $\nu = 0.98$, $\varrho = 1$ $\mu = 1$ and $\Psi(\zeta) = \zeta^{\delta}$ in Example 6.

ζ	LOM	LWOMM	LWPT	PLSM
0.1	1.0082686953	1.008069307	1.0054326	1.006641096
0.2	1.01984598096	1.019479961	1.0176608	1.018844188
0.3	1.03473185678	1.034231959	1.0351102	1.035502792
0.4	1.05292632281	1.052325304	1.0562064	1.055510424
0.5	1.07442937905	1.073759995	1.0793750	1.077760600
0.6	1.09924102552	1.098536032	1.1030416	1.101146836
0.7	1.12736126220	1.126653415	1.1256318	1.124562648
0.8	1.15879008910	1.158112143	1.1455712	1.146901552
0.9	1.19352750622	1.192912217	1.1612854	1.167057064

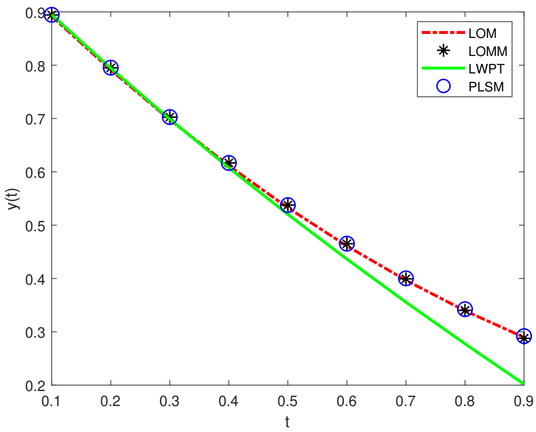

Figure 11. In Example 6, the approximate solutions of g(ζ) obtained using LOM are compared with those of other numerical methods.

6. Conclusion

In the present work, introduce a generalized Caputo-type fractional derivative with properties similar to those of the Caputo derivative. Some useful features of the proposed generalized derivative operator were discussed, including the relationship between the generalized fractional integral operator and our generalized fractional derivative operator. This type of generalized derivative seems closer to ordinary derivatives than other generalized derivatives.

Therefore, a numerical approach based on the OM of FFDEs using a new type of orthogonal polynomial to solve a class of equations that involve the fractional derivative Ψ-GC.

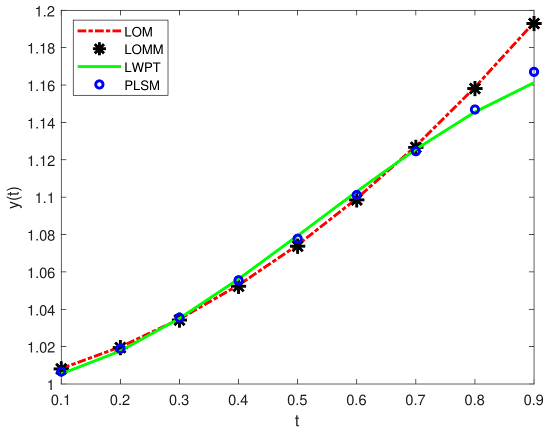

Figure 12. Example 6 presents a comparison between the approximate solutions of g(ζ) obtained using LOM and those obtained by other numerical methods.

Several examples were presented and solved to demonstrate the efficacy and precision of the proposed methodologies. Comparison of the results of this method with those of other methods highlights its effectiveness and ease of use. The final section demonstrates the high accuracy of these algorithms, showing that only a small number of Ψ-SLP and Ψ-NSLP are needed to achieve satisfactory results.

The proposed algorithm was successfully implemented to obtain precise approximate solutions and demonstrate the dynamic behaviors of the discussed systems. Therefore, it is hoped that this study will serve as a valuable tool for further implementation and investigation of generalized Caputo fractional systems.

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Conflict of interest

There is no conflict of interest to disclose.

Author contributions

All authors contributed equally to this work.

Declaration of using AI tools

The authors declare that they have not used any type of generative artificial intelligence for the writing of this manuscript, nor for the creation of images, graphics, tables, or their corresponding captions.

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